Pogojna verjetnost in Bayesov izrek

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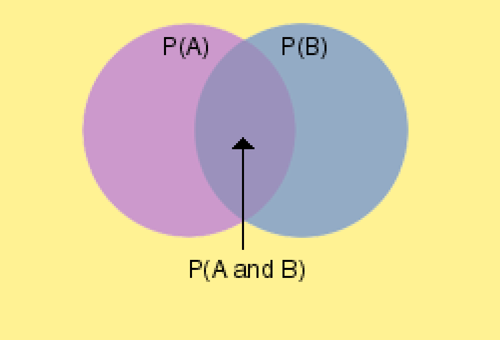
Maj, 2012

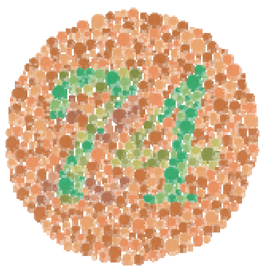
Pogojna verjetnost

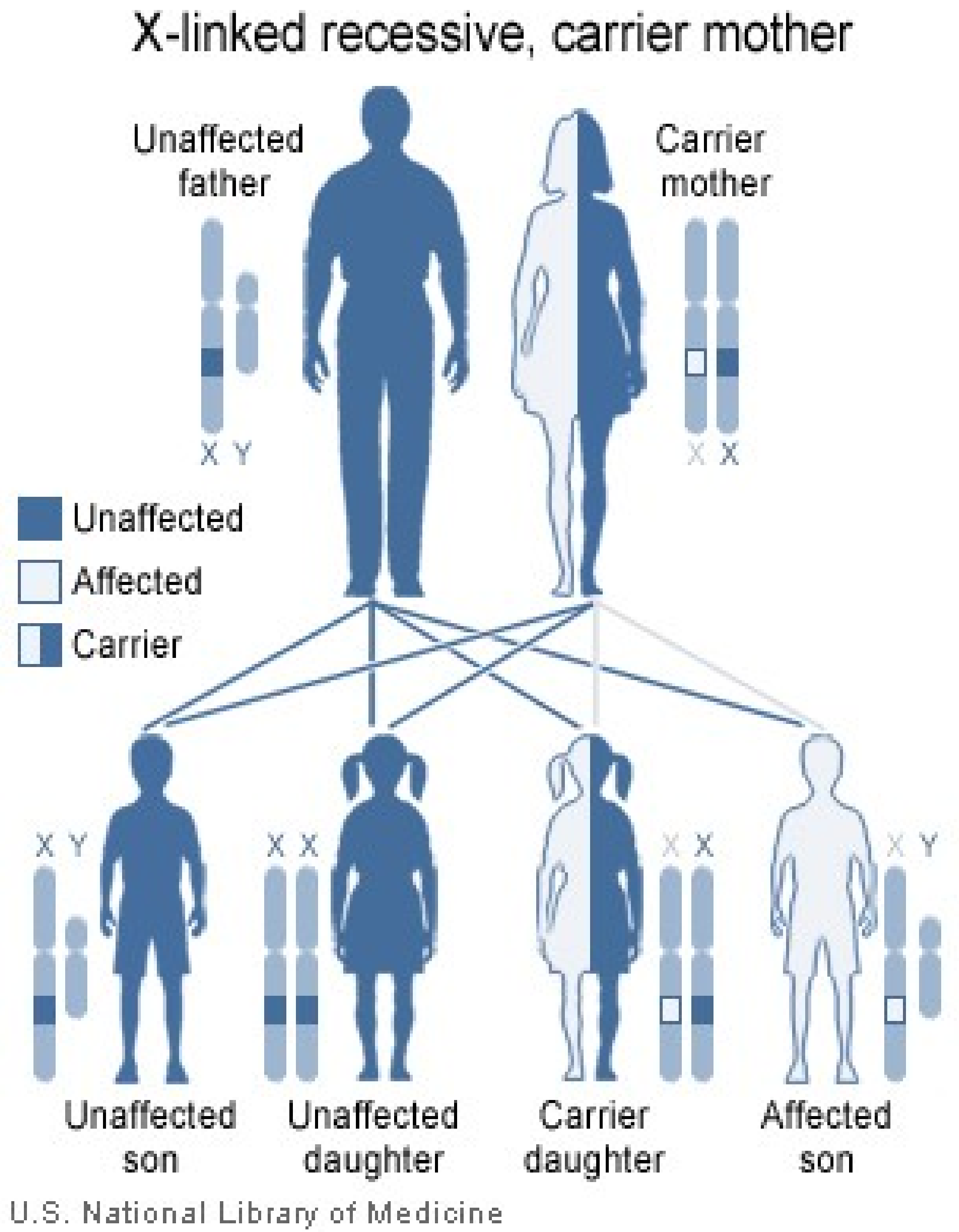
- Verjetnost dogodka “a” pri pogoju, da se je zgodil dogodek “b”
- Zapis:
  - $p(a|b)$
  - | pomeni “če je”, če imamo”, “če velja”
- Uporaba:
  - Diagnostični testi (občutljivost/specifičnost)
  - Analiza podatkov, primerjava modelov
  - Markovski procesi



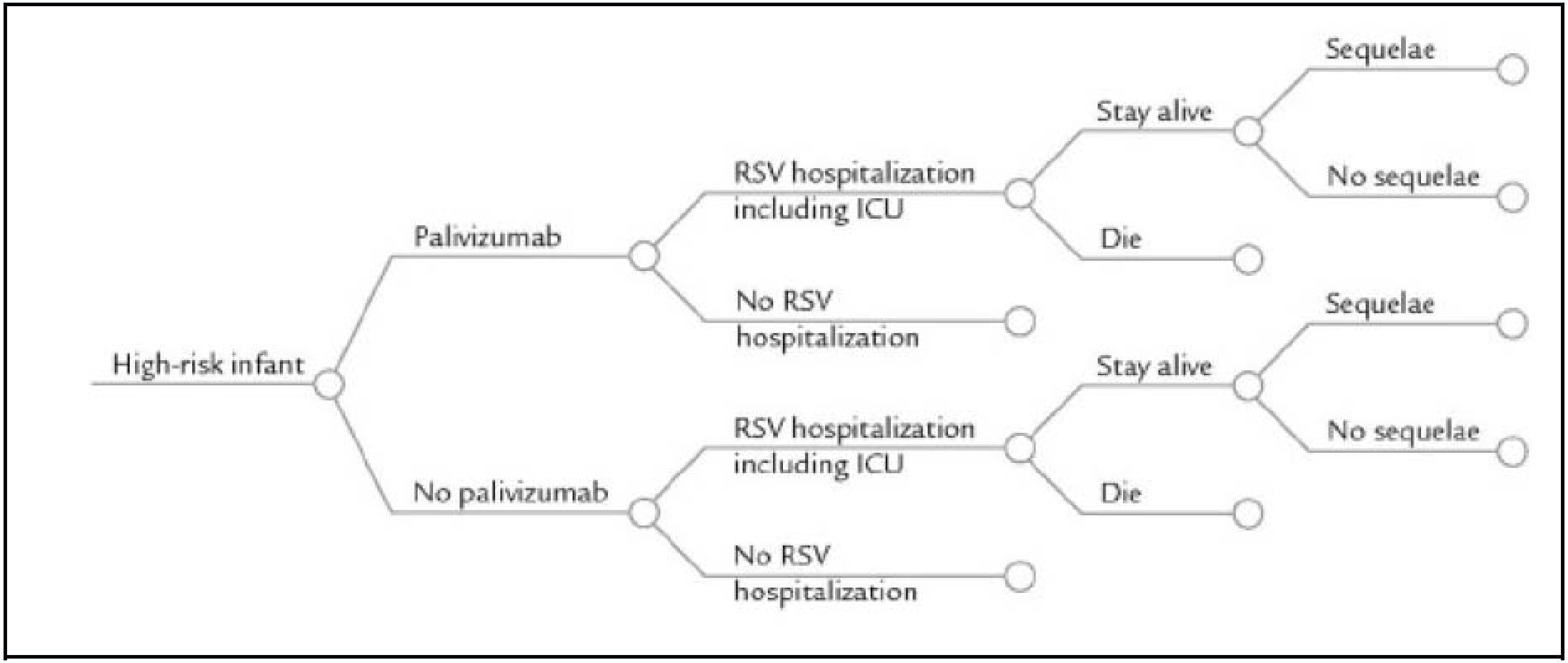
Pogojna verjetnost

- Verjetnost, da bo otrok deček ($p=0,51$)
- Verjetnost krvne skupine A ($p=0,43$)
- Kakšna je verjetnost, da bo otrok deček s krvno skupino A?
- Verjetnost barvne slepote ($p=0,04$)
- Kakšna je verjetnost, da bo otrok deklica z barvno slepoto?




X-linked recessive, carrier mother
Unaffected father
Carrier mother
X Y
X X
Unaffected
Affected
Carrier
X Y
X X
X X
X Y
Unaffected son
Unaffected daughter
Carrier daughter
Affected son
U.S. National Library of Medicine

Odločitveni model



Slika 1: Model odločitvenega drevesa za ocenjevanje stroškovne učinkovitosti palivizumaba (ang.)

Igralne karte

♠	♣	♥	♦
13	13	13	13
13	13	7	13

P(S)

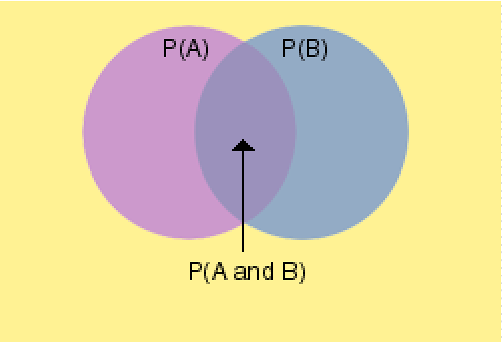
P(F)

P(S|F)

P(F|S)

Primer pogojne verjetnosti

- Diagnostični klinični test
- Dogodka
  - Dogodek A: Pacient ima bolezen
  - Dogodek B: Test je pozitiven



- Zapis:

$p(A \cap B)$ — Verjetnost, da ima pacient bolezen in pozitivni izid testa

$p(A \cap B')$ — Verjetnost, da ima pacient bolezen, a je izid testa negativen (lažno negativen)

$p(A' \cap B)$ — Verjetnost, da pacient nima bolezni, a je izid testa pozitiven (lažno pozitiven)

$p(A|B)$ — Verjetnost, da ima pacient bolezen, če je izid testa pozitiven

$p(A|B')$ — Verjetnost, da ima pacient bolezen, če je izid testa negativen

30% pacientov, ki so bili v letu 2008 sprejeti v enoti nujne medicinske pomoči UKC Ljubljana zaradi bolečine v trebušni votlini, je imelo akutno vnetje slepiča. 70% bolnikov z vnetjem slepiča je imelo telesno temperaturo nad 37,5°C. Pri pacientih, ki niso imeli vnetja slepiča, pa je bila telesna temperatura višja od 37,5°C pri 40% bolnikov.

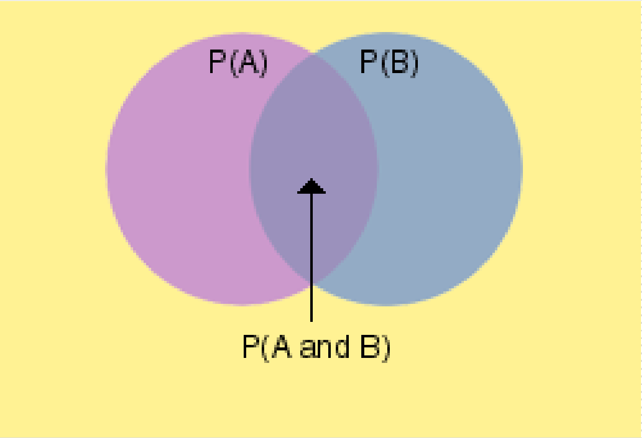
- Kaj lahko rečemo o dogodku A, če poznamo izid B?
  - Pozitiven test = bolezen, negativen test = zdrav?
  - Diagnostični testi niso idealni!

$$p(A \cap B) = p(A|B)p(B) = p(B|A)p(A)$$

$$p(A|B') = \frac{p(A \cap B')}{p(B')} = \frac{p(A \cap B')}{1 - p(B)}$$

$$p(A|B) = \frac{p(B|A)p(A)}{p(B)} = \frac{p(B|A)p(A)}{p(B|A)p(A) + p(B|A')p(A')}$$

- Odvisnost dogodkov A in B
  - Ali je $p(A)$ odvisna od $p(B)$?
  - Ali je $p(B)$ odvisna od $p(A)$?



Bayesov teorem

Thomas Bayes (1702-1761)

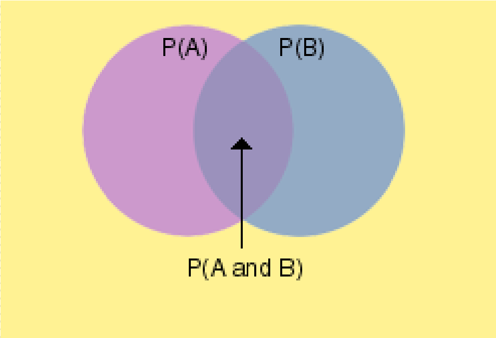
- Dogodki A_i kjer je i = 1 do k so:
  - Nezdružljivi: $A_i \cap A_j = 0$ za vse i,j

$$P(A_1 \cup \cdots \cup A_k) = 1$$

- Za vsak dogodek B

$$p(B) = p(B|A_1)p(A_1) + \cdots + p(B|A_k)p(A_k)$$

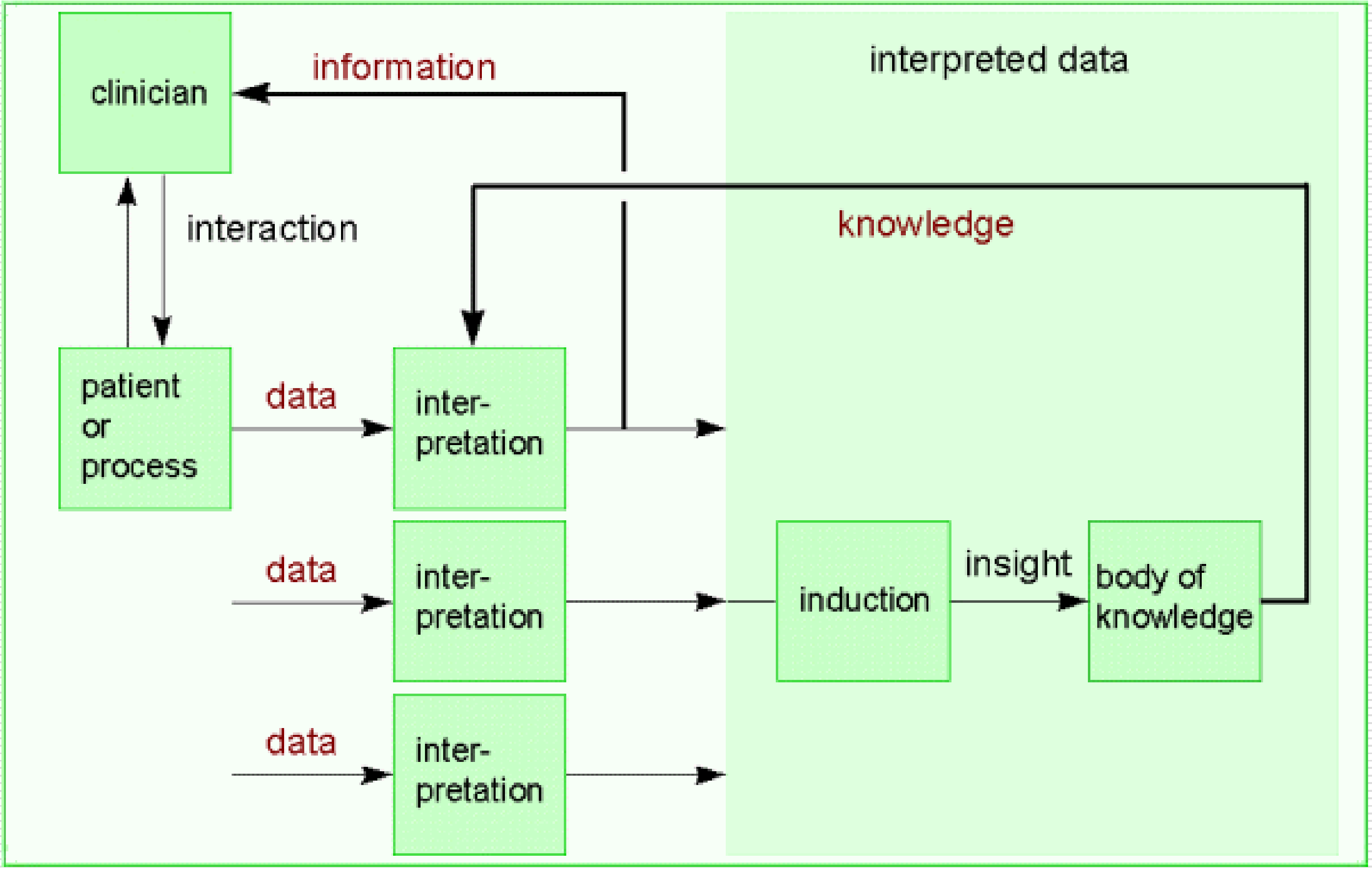
$$p(B) = \sum_{i=1}^{k} p(B|A_i)p(A_i)$$



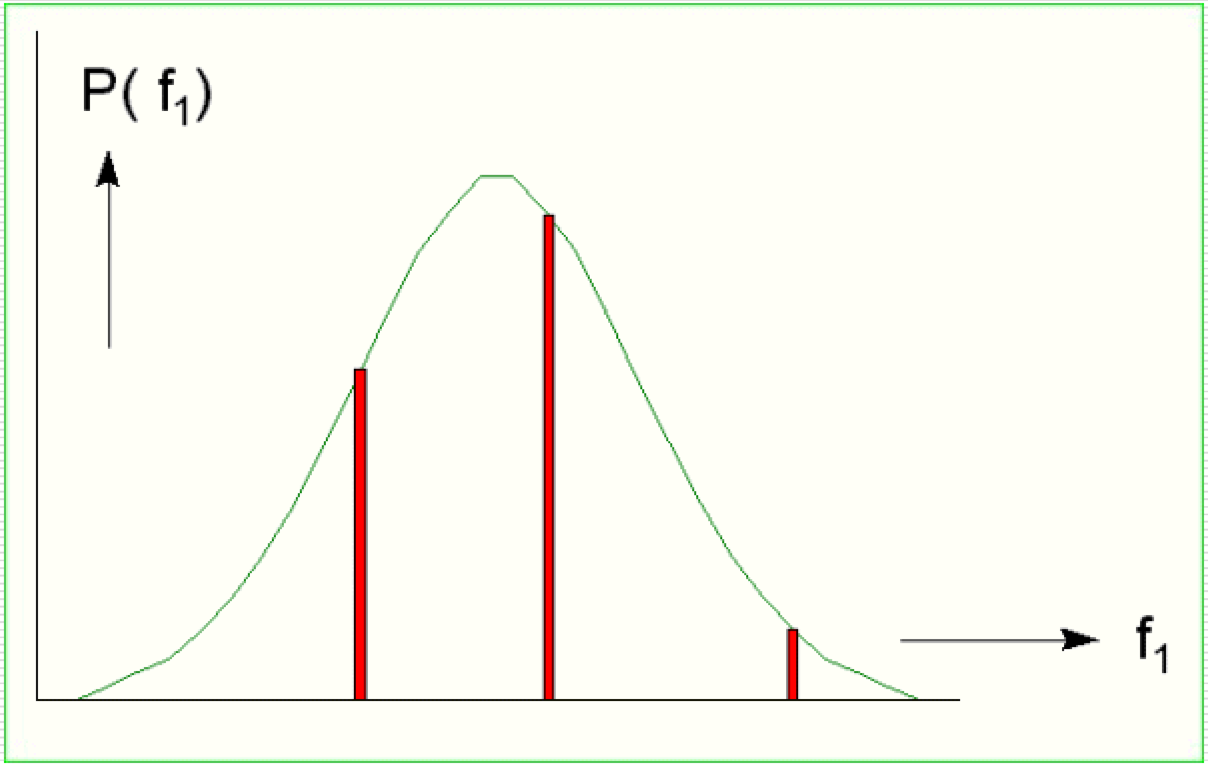
- Bayesov teorem

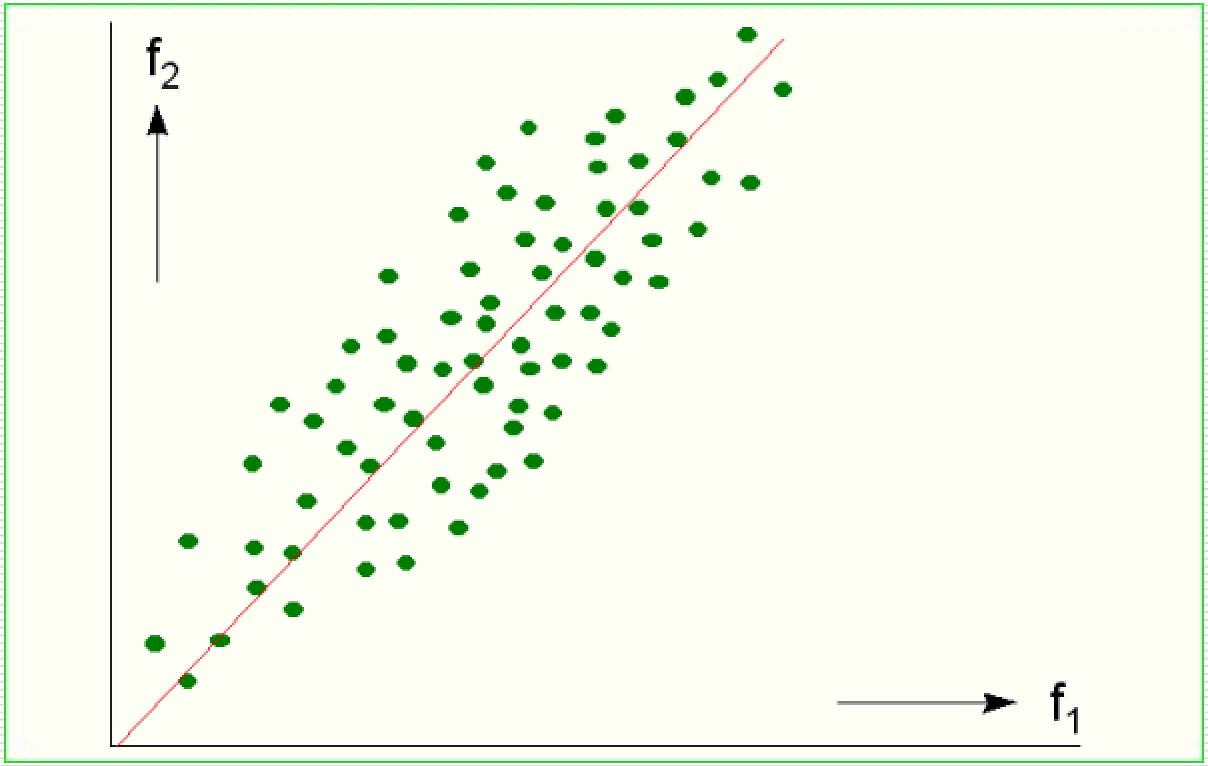
$$p(A_j|B) = \frac{p(A_j \cap B)}{p(B)} = \frac{p(B|A_j)\cdot p(A)}{\sum_{i=1}^{k} p(B|A_i)p(A_i)}$$

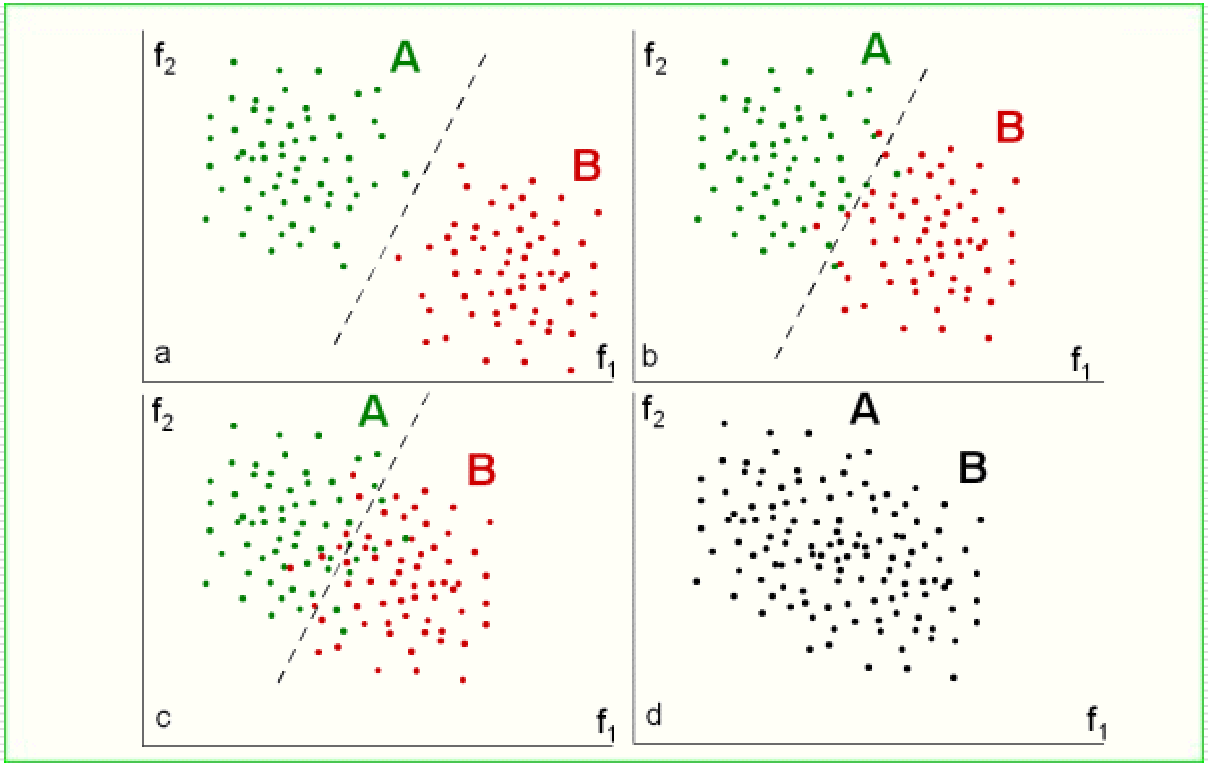
Podatek, informacija in znanje



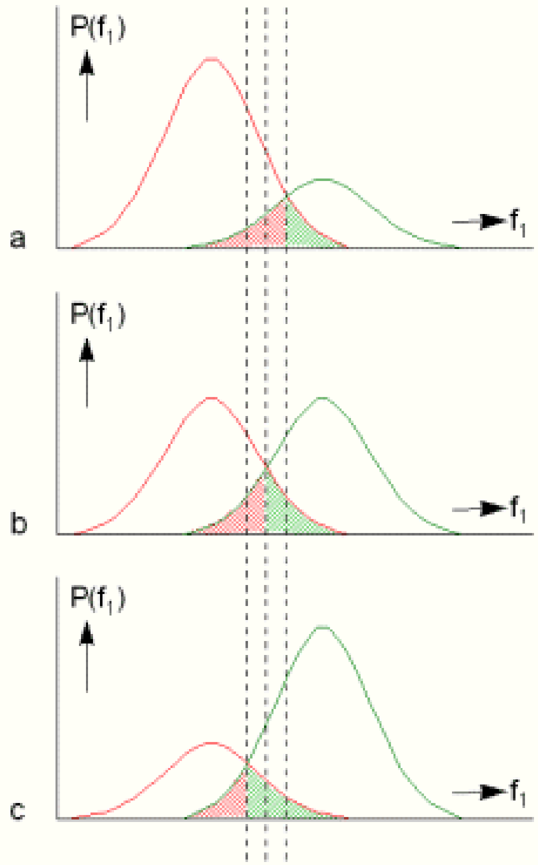
Učenje




f2
f1



f2
A
B
a
f1
f2
A
B
b
f1
f2
A
B
c
f1
f2
A
B
d
f1


P(f_1)
a
f_1
P(f_1)
b
f_1
P(f_1)
c
f_1

15.01 Relationships between True Positive (TP), True Negative (TN), False Positive (FP), and False Negative (FN)

		Decision Model		
		+	-	
Truth	+	TP	FN	100%
	-	FP	TN	100%

Table 15.1. Relationships between True Positive (TP), True Negative (TN), False Positive (FP), and False Negative (FN).

Občutljivost in specifičnost

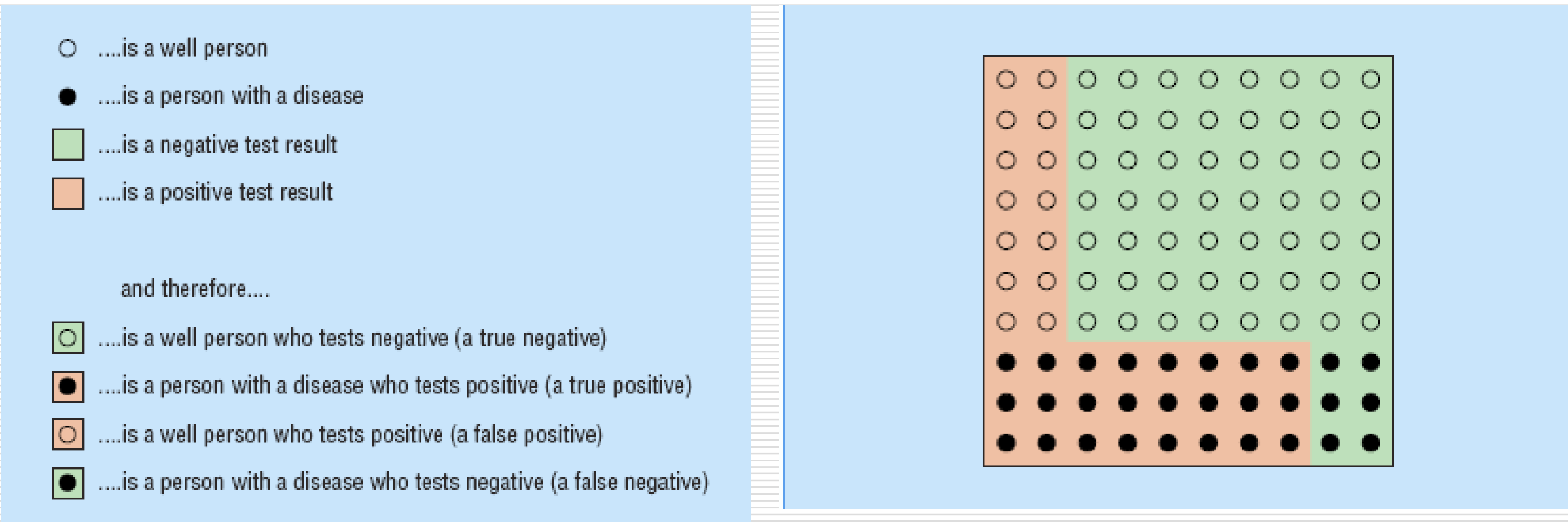
15.02 Illustration of Sensitivity, Specificity, and Predictive Value

		Decision Model +	Decision Model -	
Truth	+	a	b	a + b
	-	c	d	c + d
		a + c	b + d	a + b + c + d

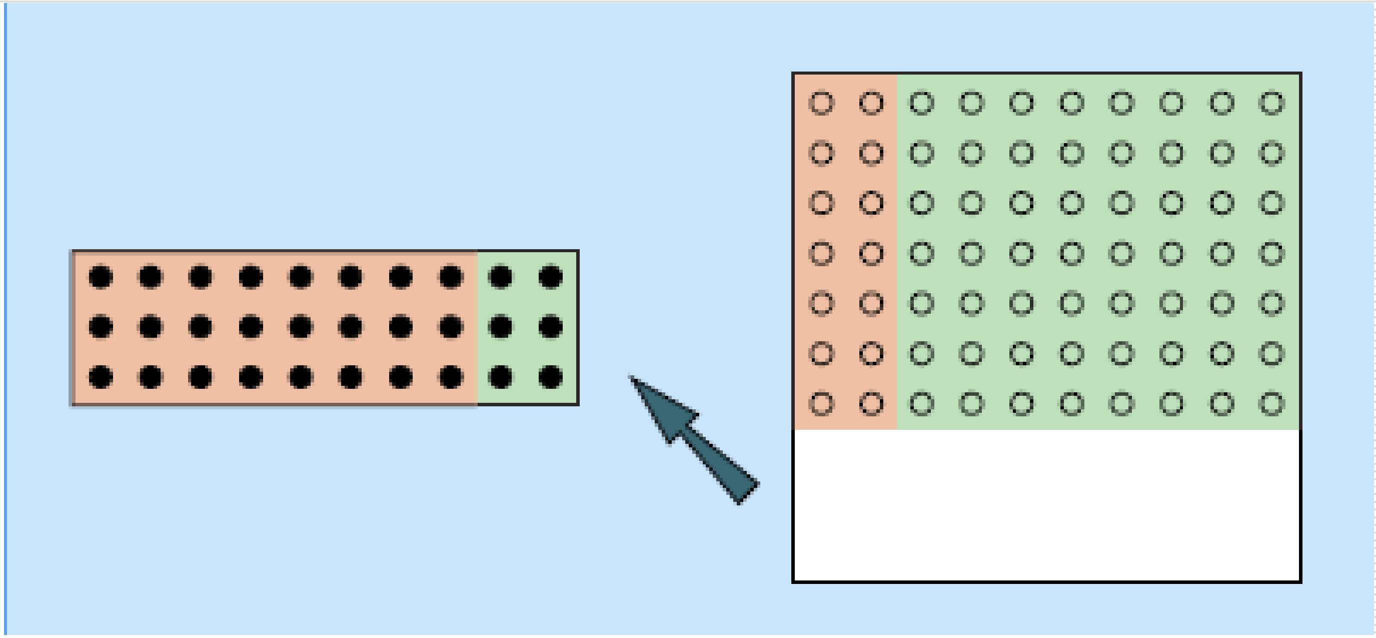
Table 15.2. Illustration of Sensitivity, Specificity, and Predictive Value (see text).

Občutljivost TP=a/(a+b)
Specifičnost TN=d/(c+d)
Pozitivna napovedna vrednost PNV=a/(a+c)
Negativna napovedna vrednost NNV=d/(b+d)

Understanding sensitivity and specificity with the right side of the brain

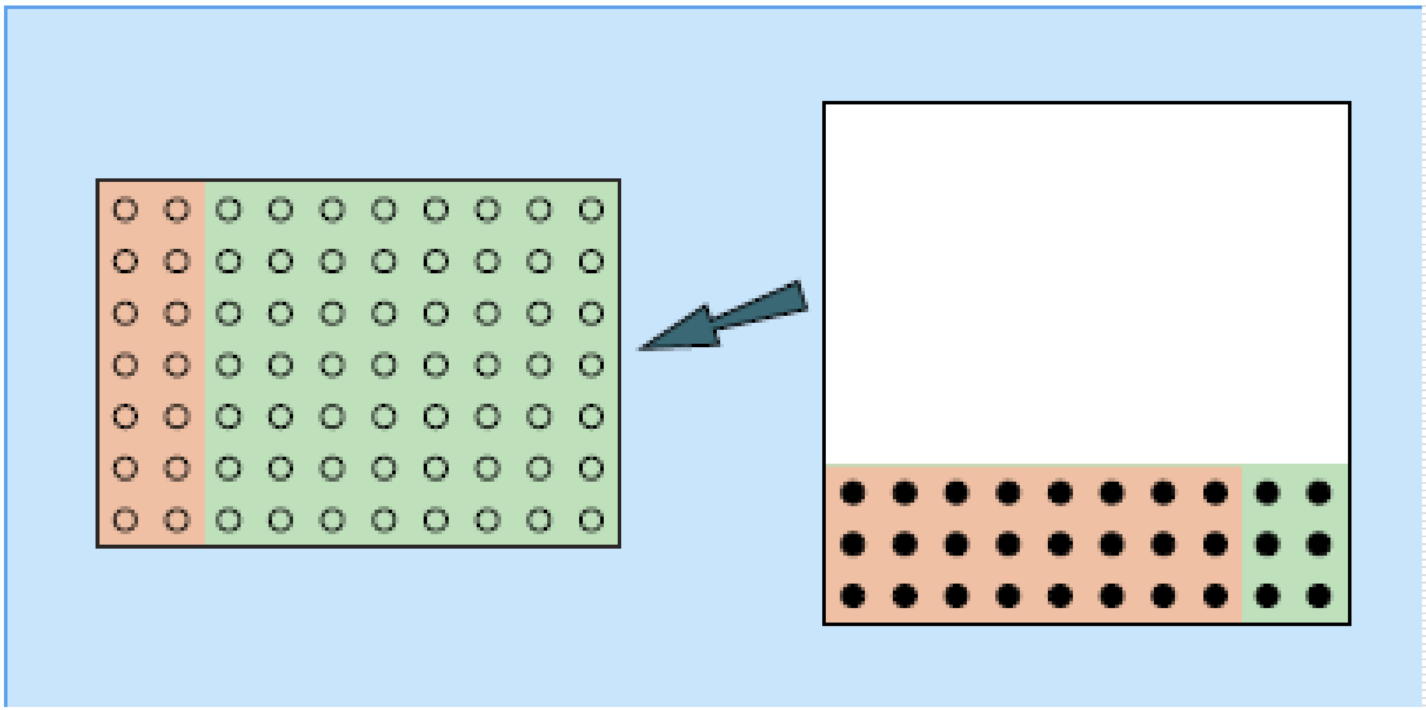


Občutljivost



24/30=80%

Specifičnost



56/70=80%

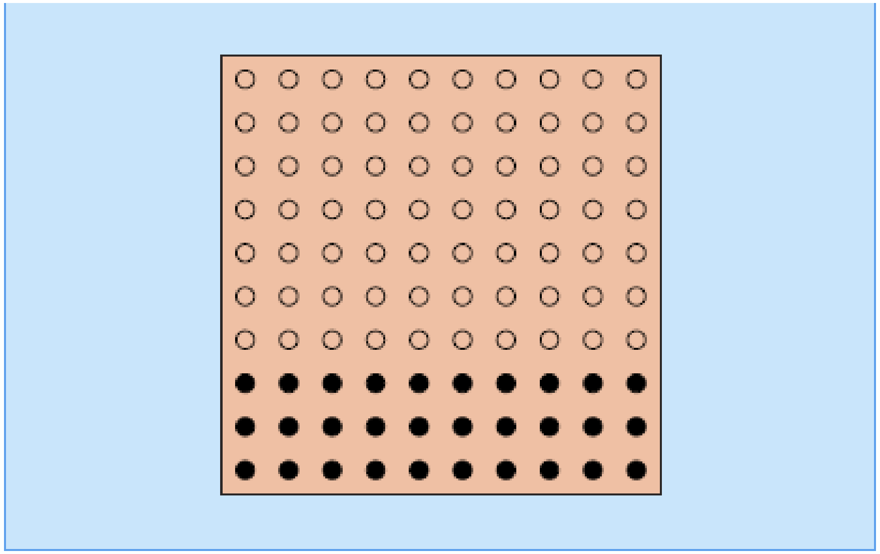


Fig 6 Test with 100% sensitivity

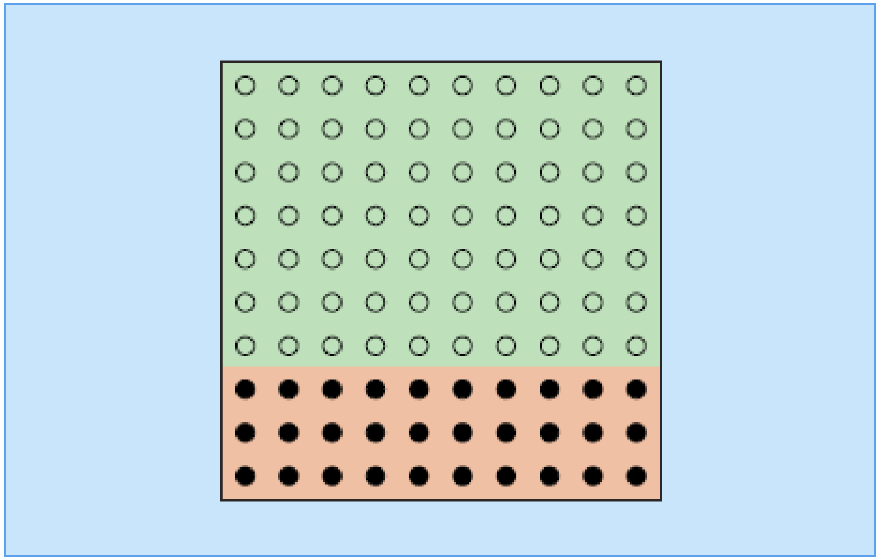
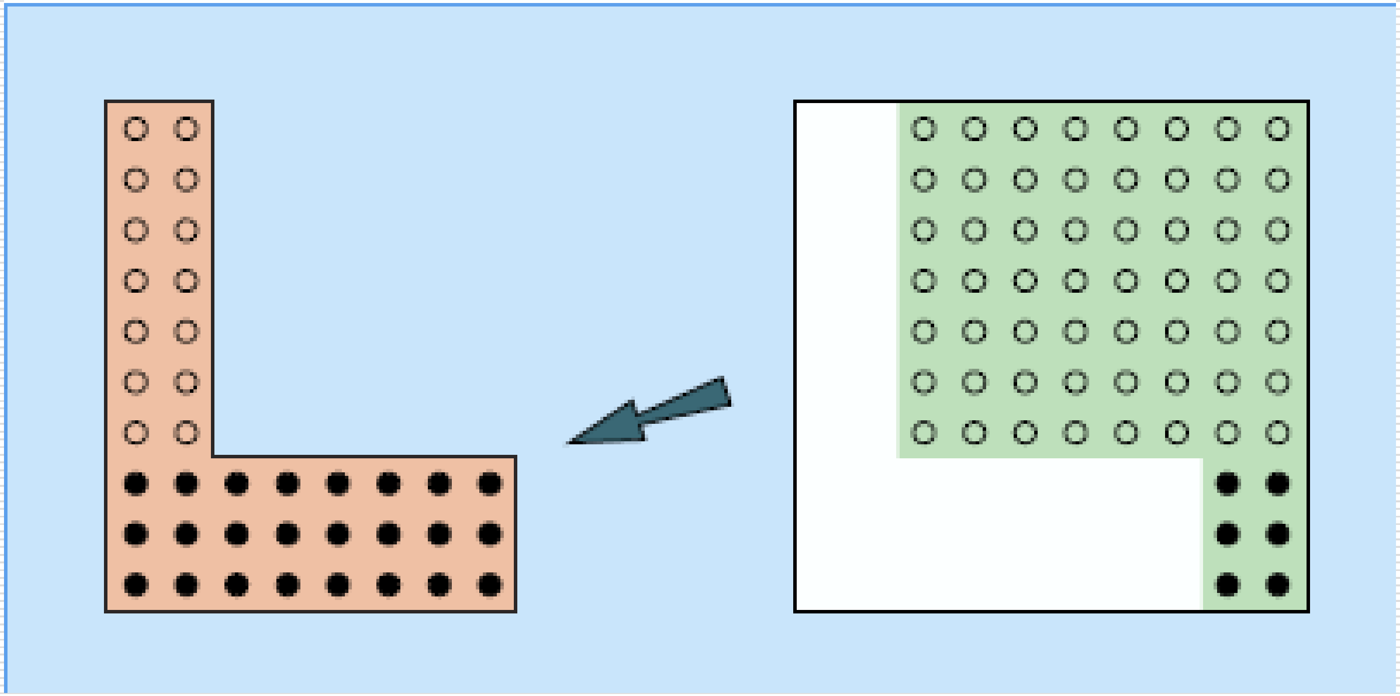


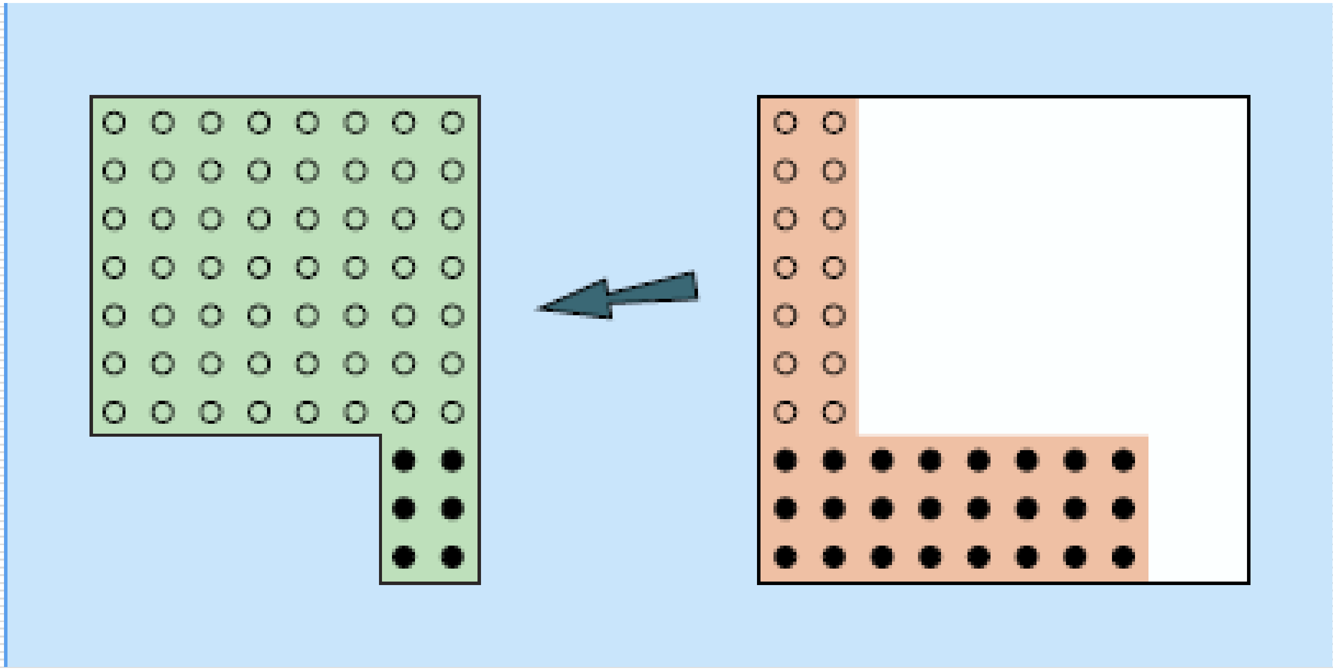
Fig 7 Perfect test

Pozitivna napovedna vrednost

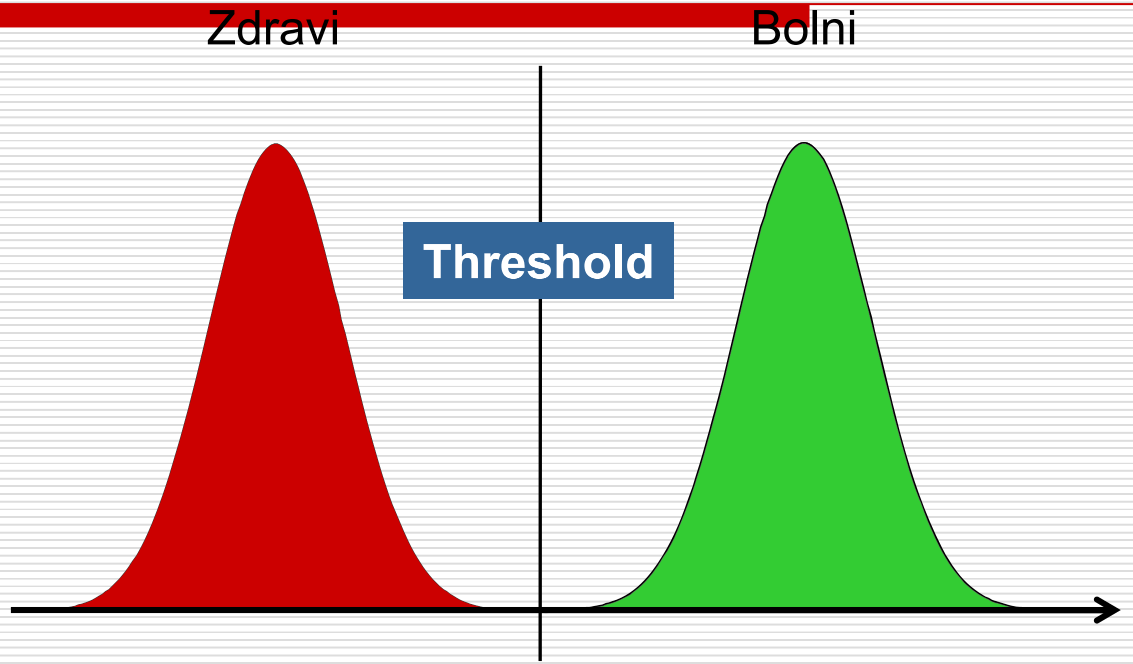


24/38=63%

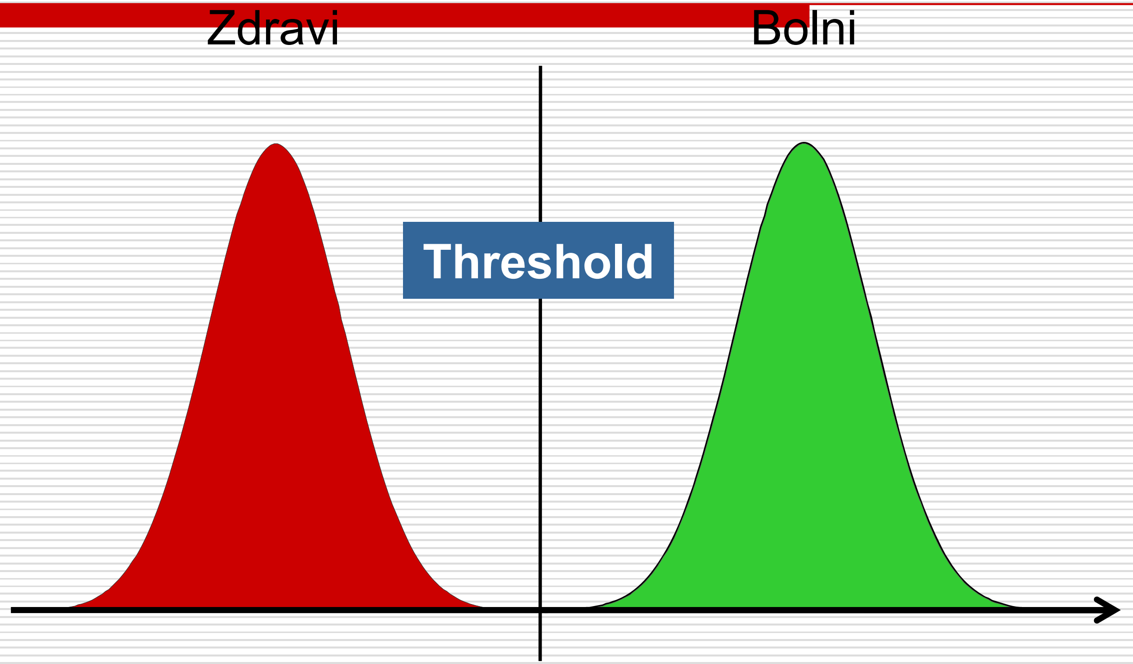
Negativna napovedna vrednost



56/62=90%



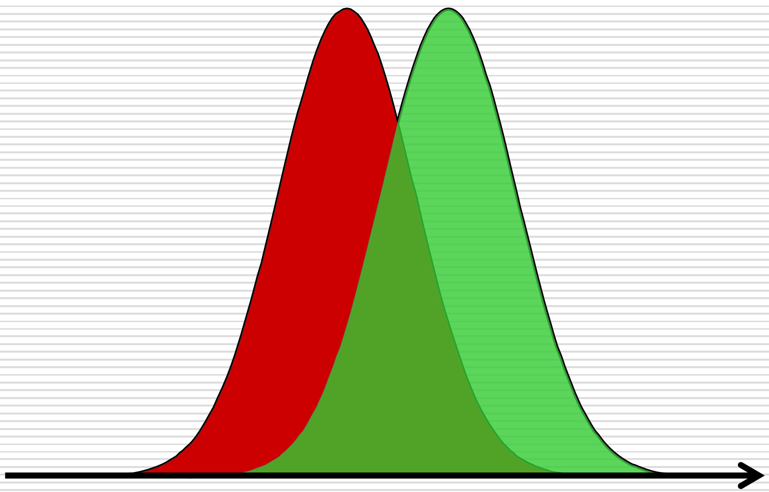
Vrednost diagnostičnega parametra
ali
Subjektivna ocena verjetja bolezni



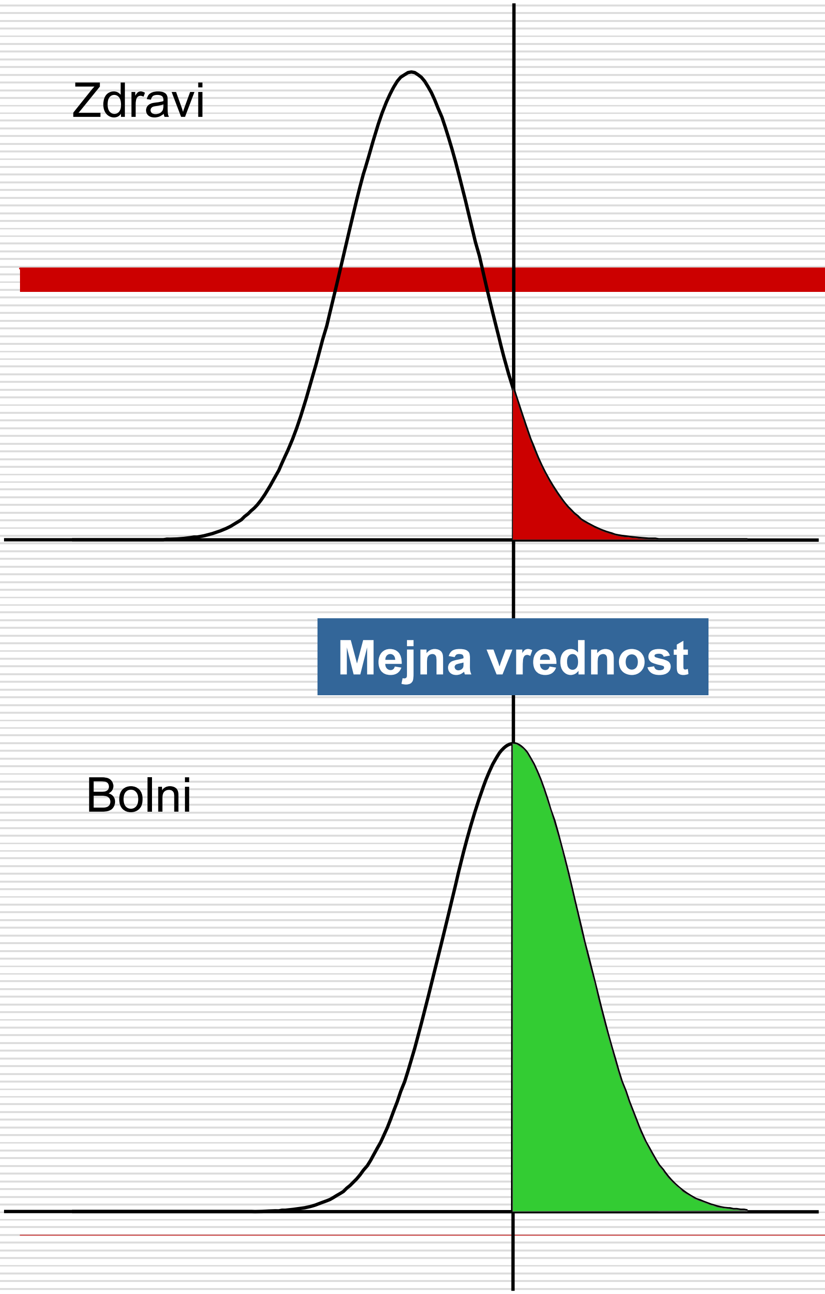
Vrednost diagnostičnega parametra
ali
Subjektivna ocena verjetja bolezni

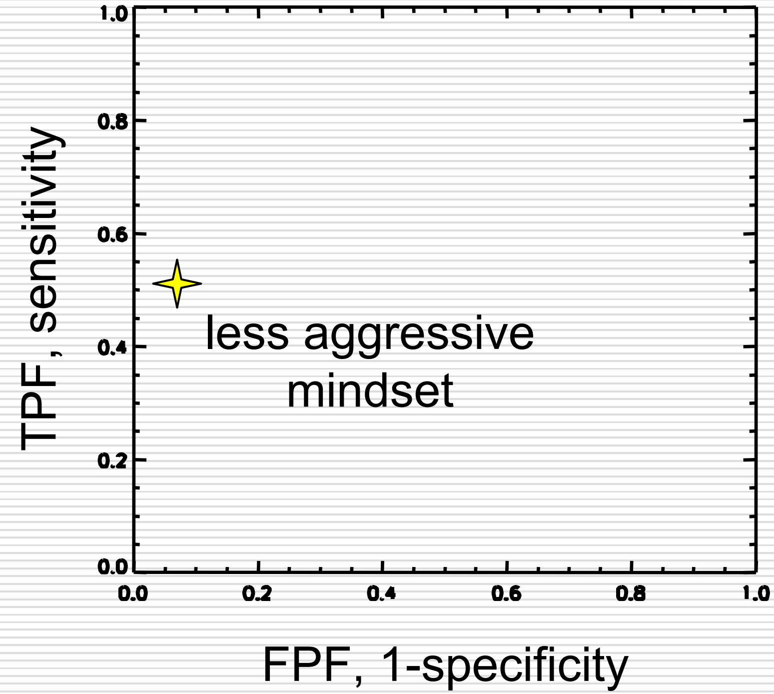
Zdravi Bolni

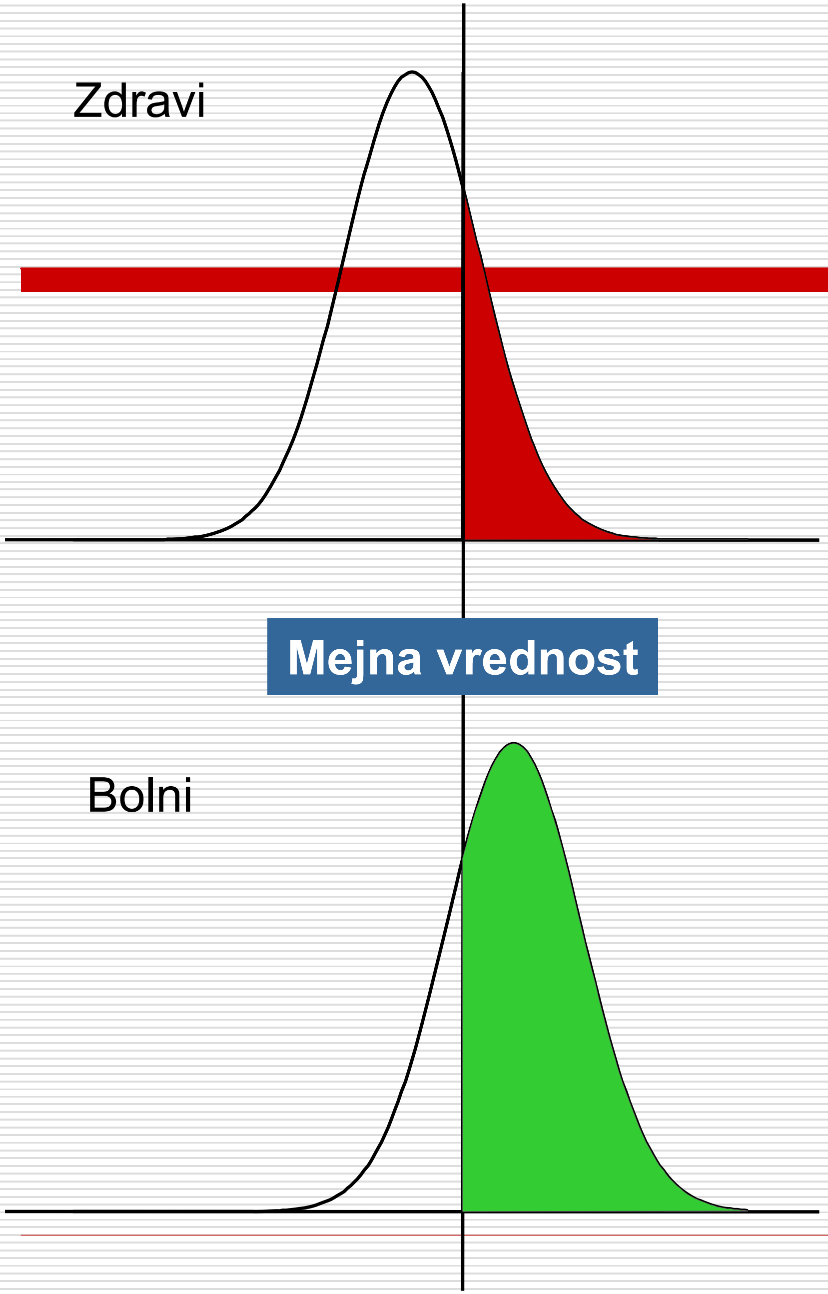
Običajneje:

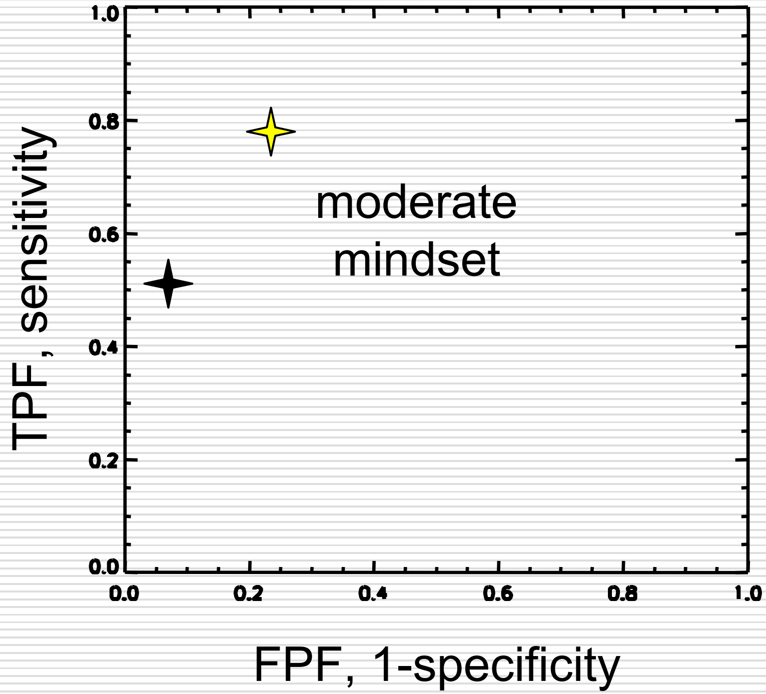


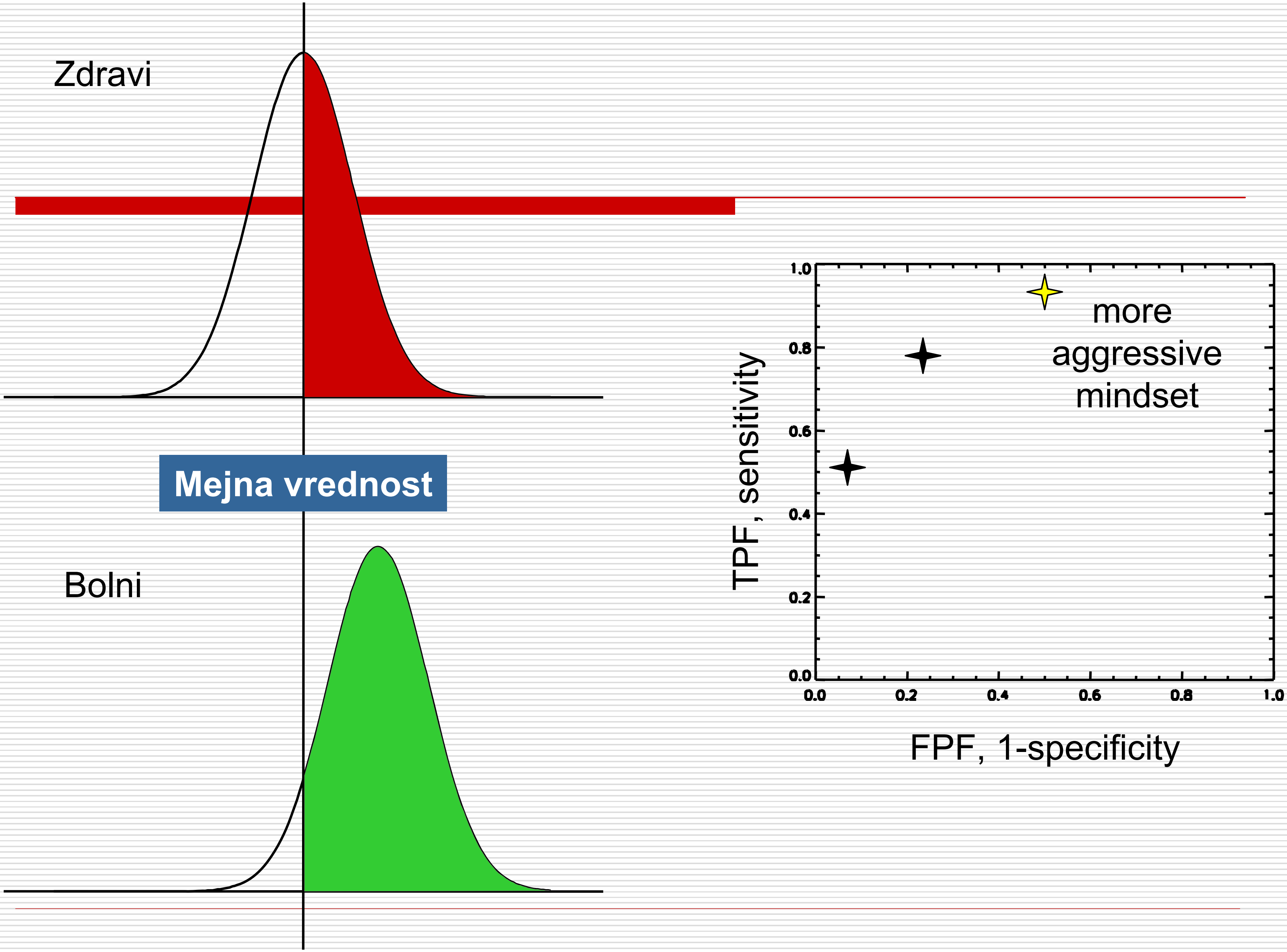
Vrednost diagnostičnega parametra
ali
Subjektivna ocena verjetja bolezni

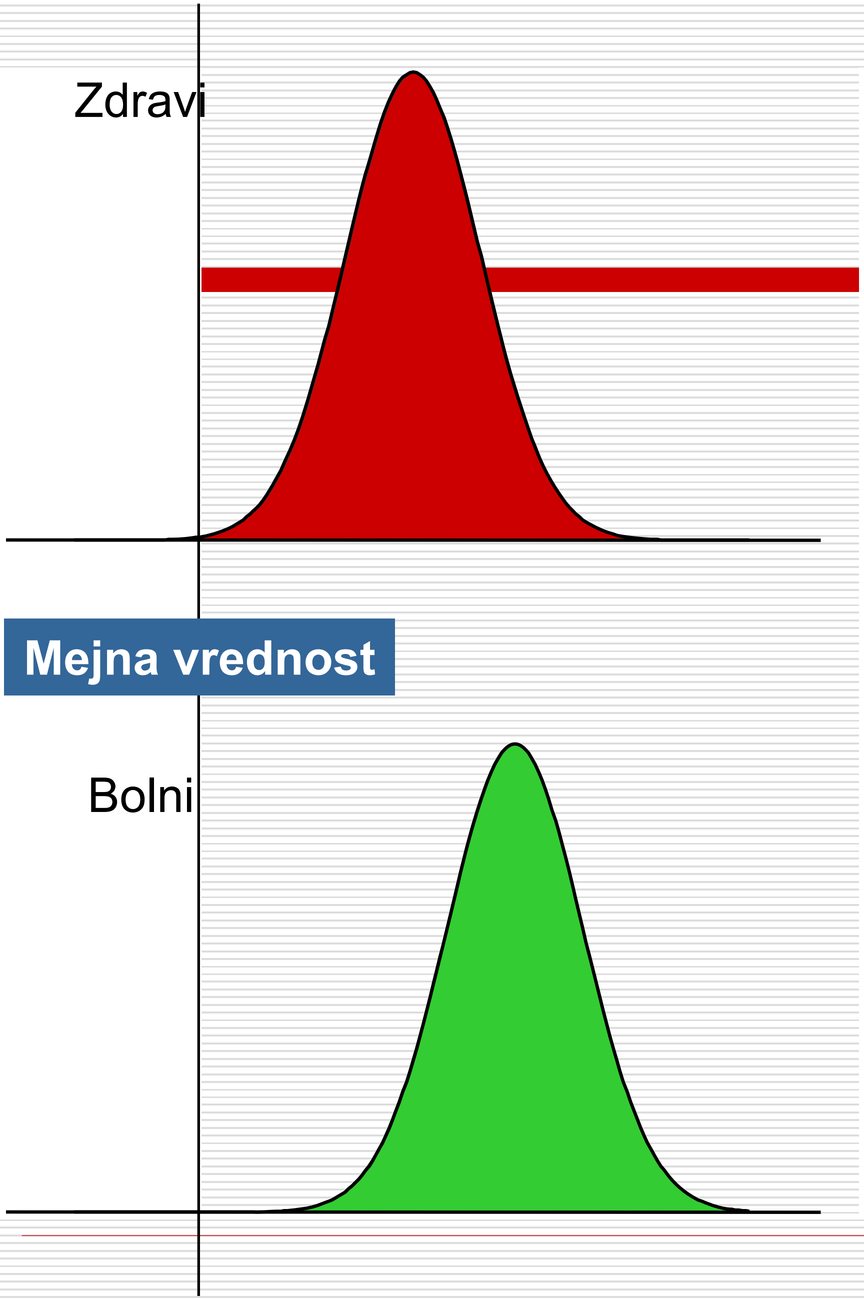

Zdravi
Mejna vrednost
Bolni


TPF, sensitivity
FPF, 1-specificity
less aggressive mindset
0.0
0.2
0.4
0.6
0.8
1.0

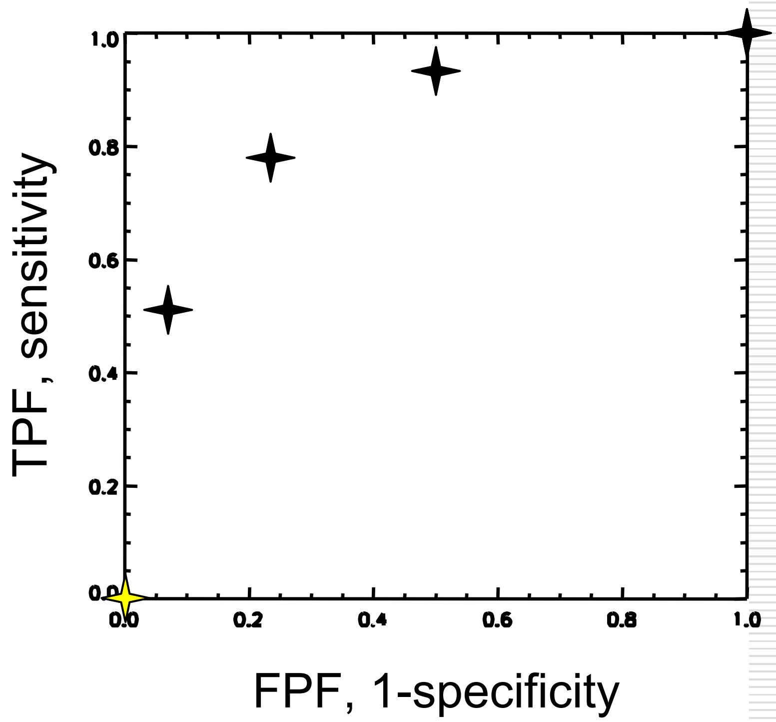

Zdravi
Mejna vrednost
Bolni


TPF, sensitivity
FPF, 1-specificity
moderate
mindset
0.0
0.2
0.4
0.6
0.8
1.0

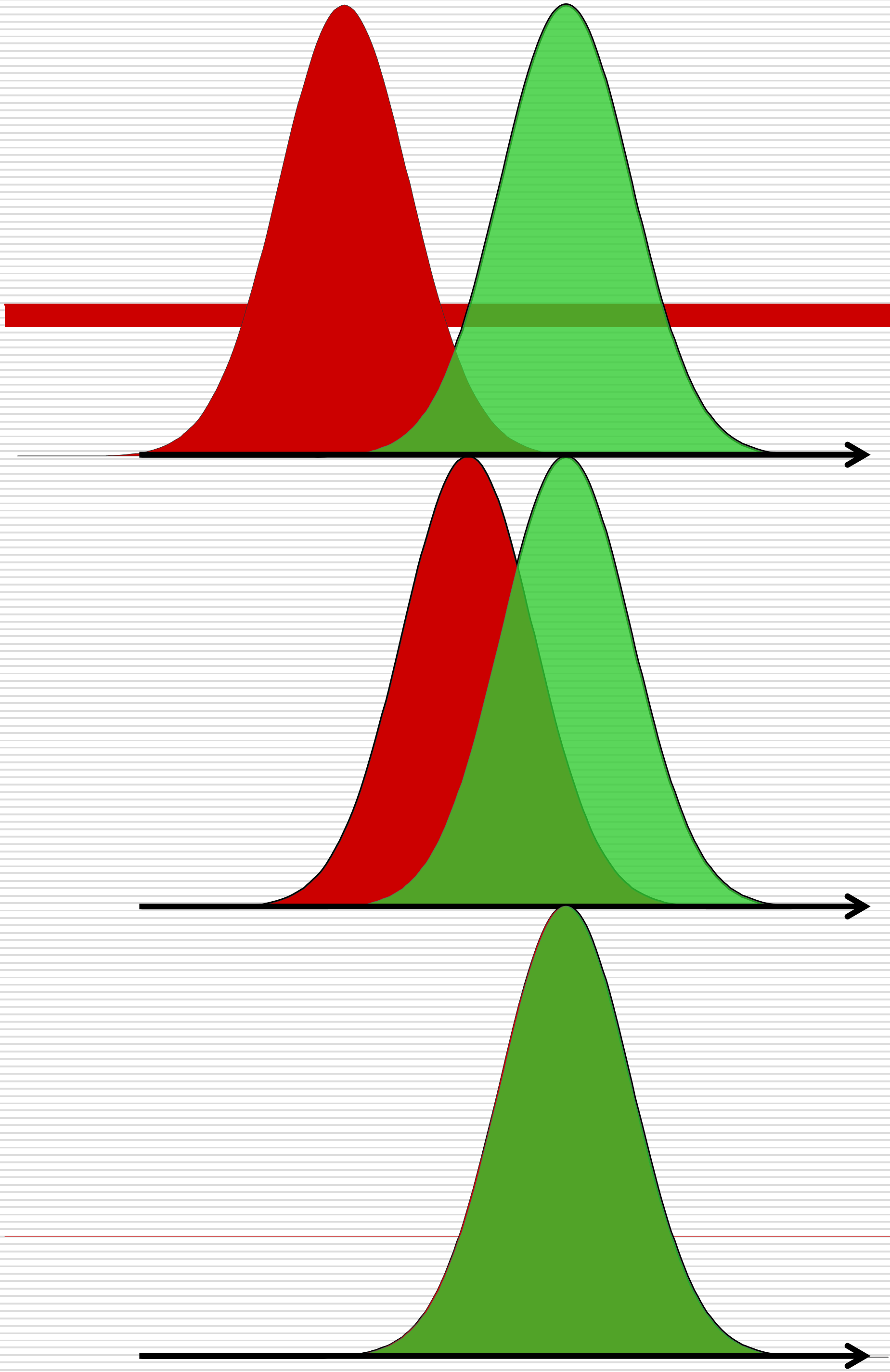

Zdravi
Mejna vrednost
Bolni
1.0
0.8
0.6
0.4
0.2
0.0
TPF, sensitivity
0.0
0.2
0.4
0.6
0.8
1.0
FPF, 1-specificity
more
aggressive
mindset

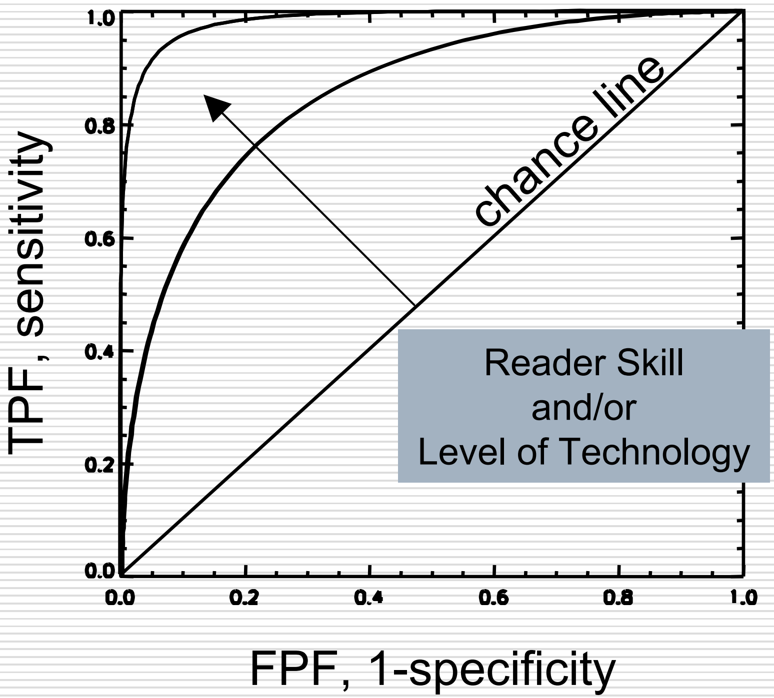


Entire ROC curve



Entire ROC curve





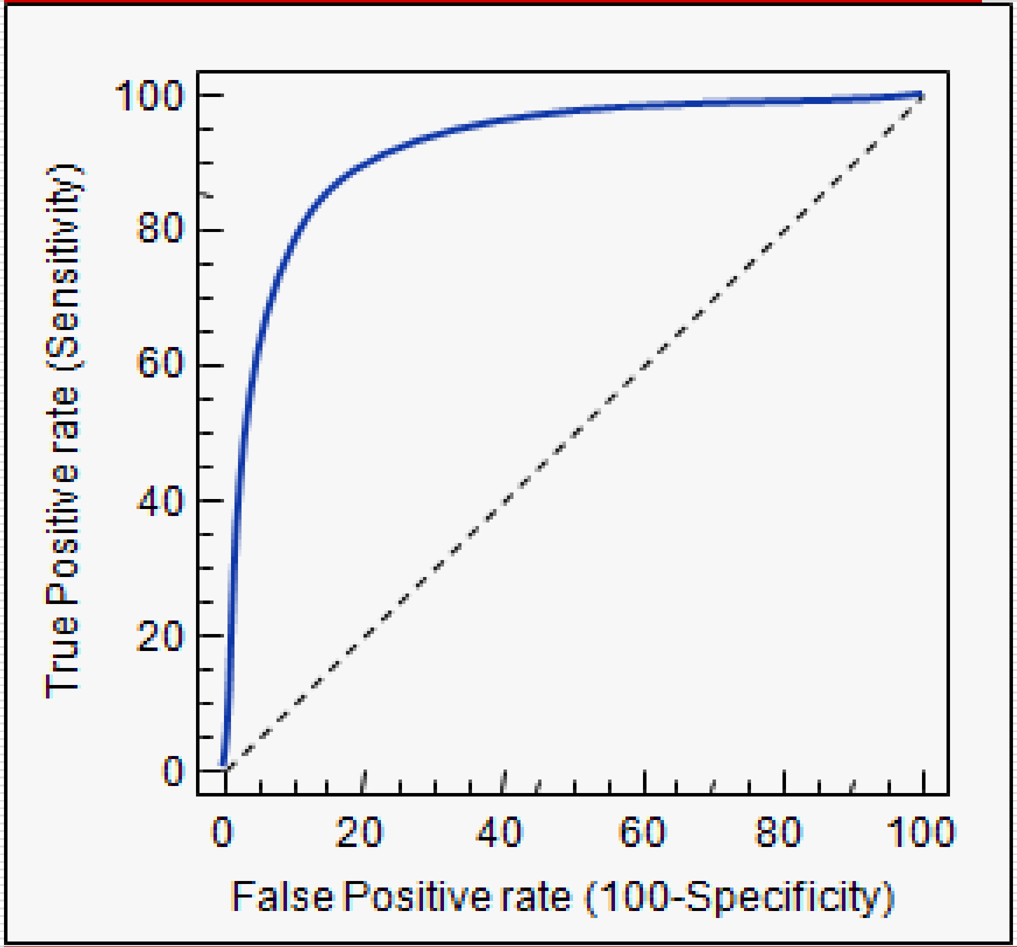
ROC

Receiver Operating Characteristic
(historično ime iz raziskav z radarjem)

Relative Operating Characteristic

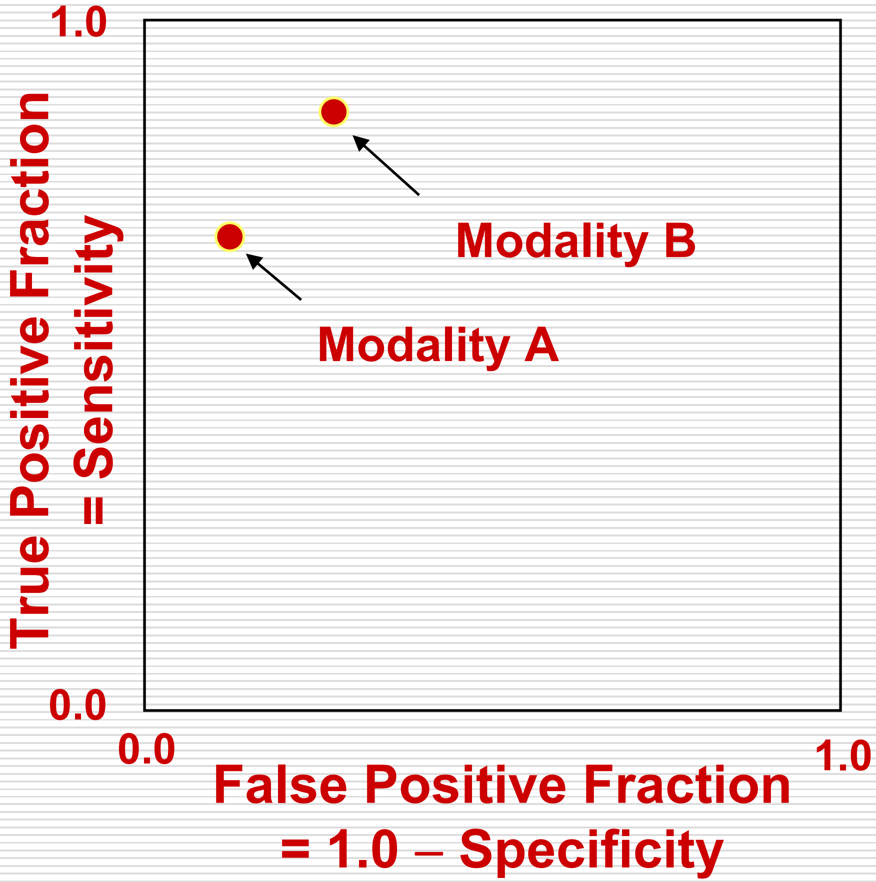
Operating Characteristic

ROC
(Relative operating characteristics)

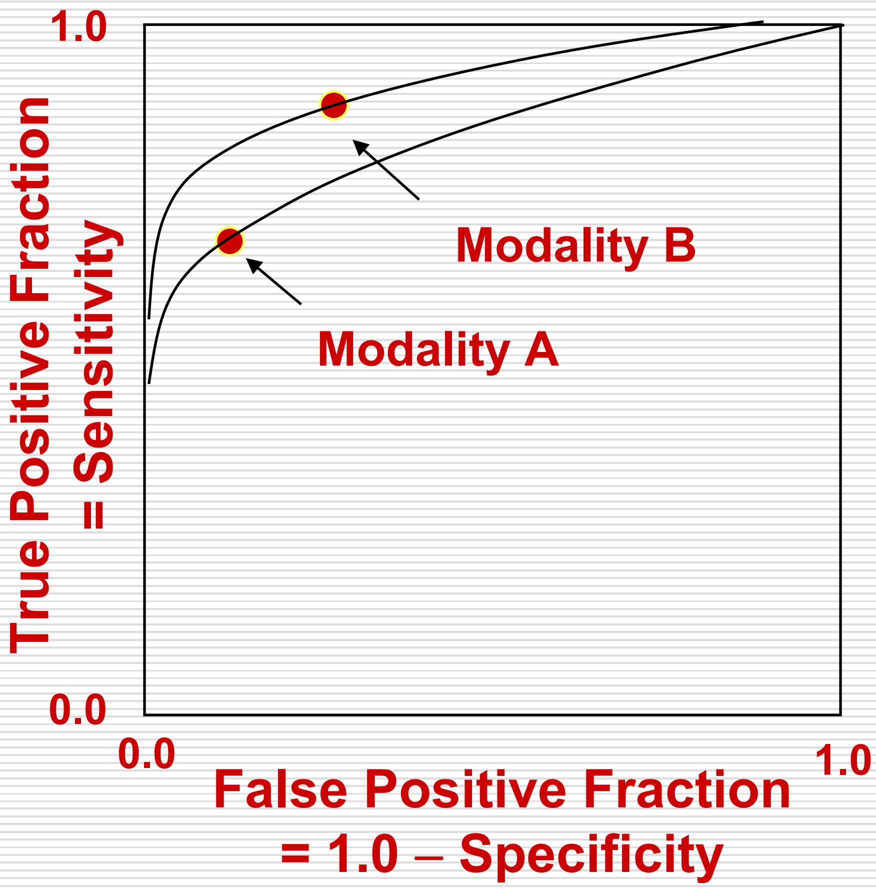


Nekaj ROC krivulj . . .

Kateri diagnostični test je boljši?



Za odgovor moramo poznati ROC krivulji (prva možnost):

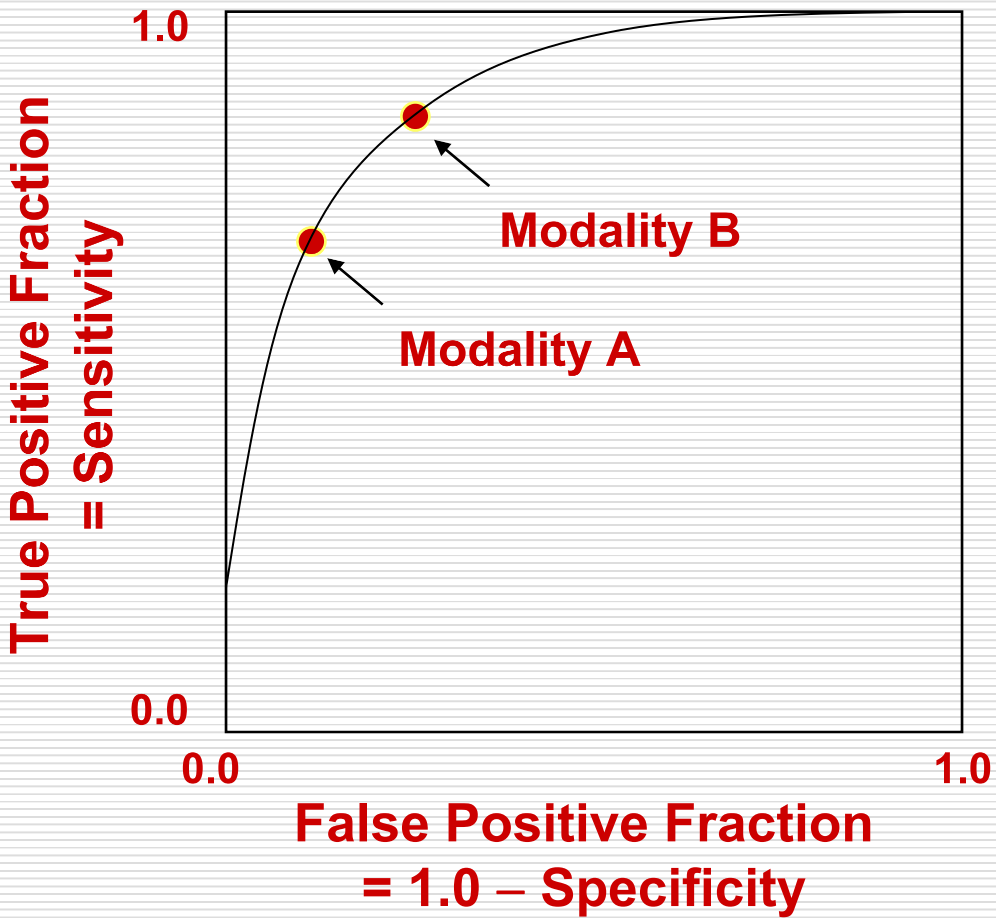


B je boljši:

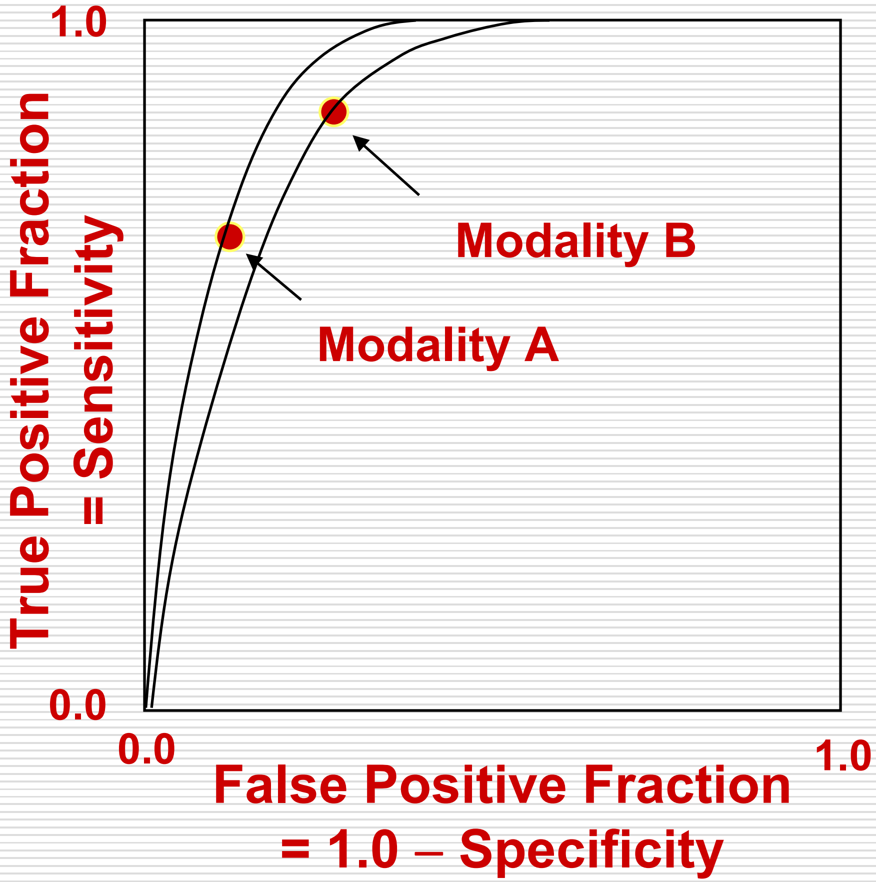
večji TPF pri istem FPF, ali

manjši FPF pri istem TPF

Druga možnost: Ista ROC



. . . In še tretja:

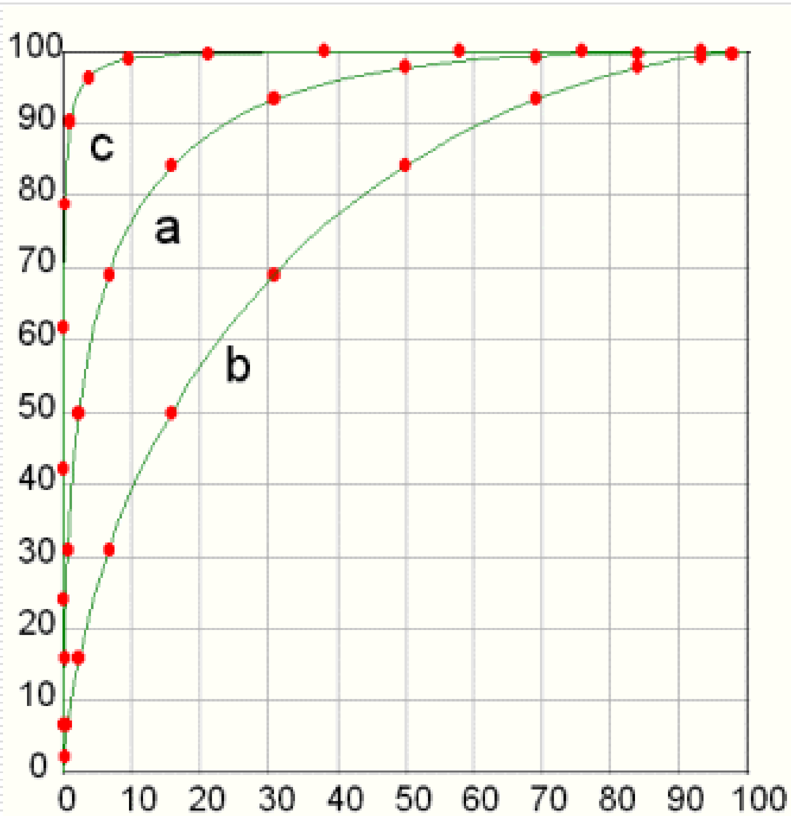


<u>**A je boljši:**</u>

večji TPF pri istem FPF, <u>ali</u>

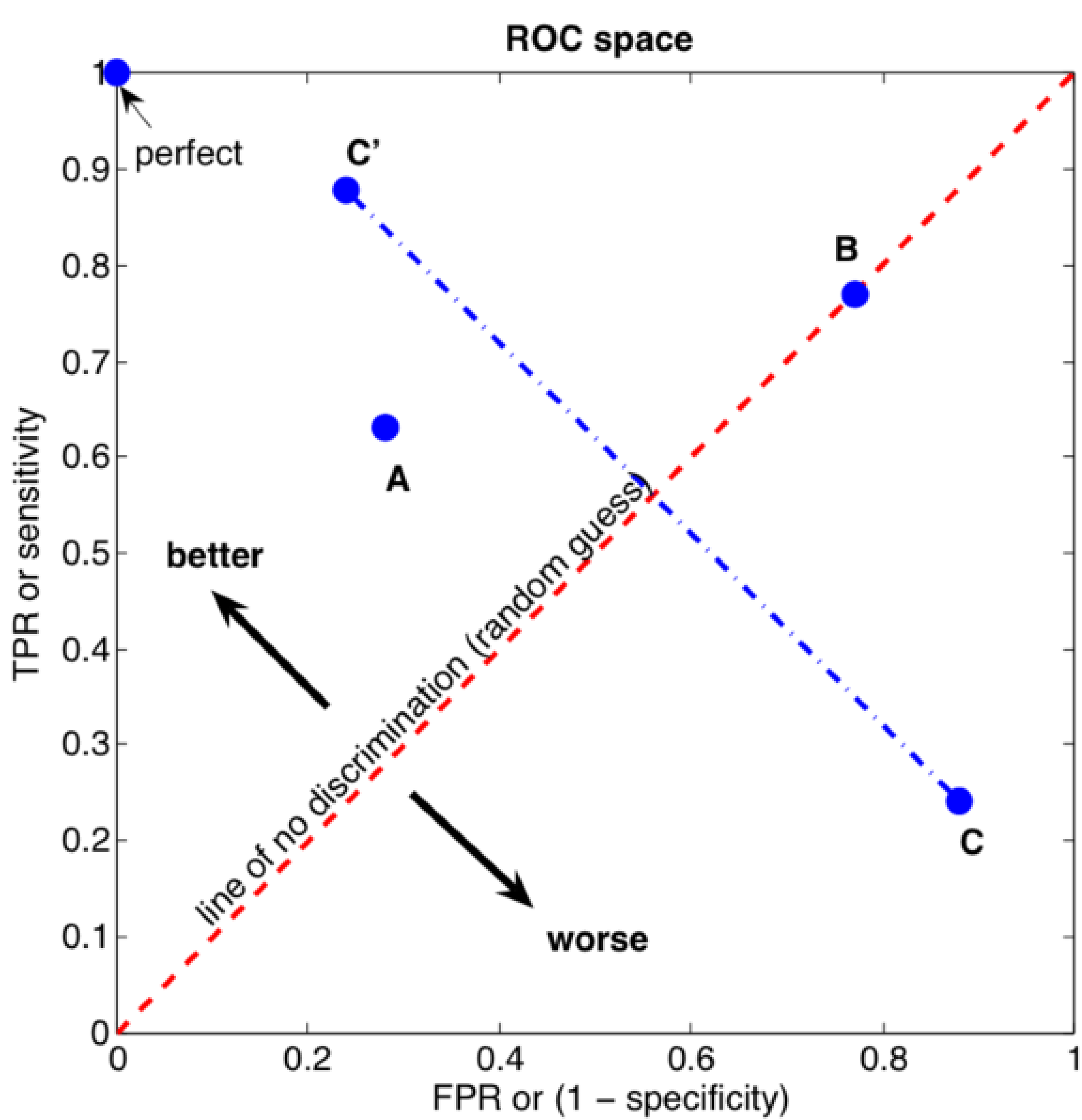
manjši FPF pri istem TPF

ROC



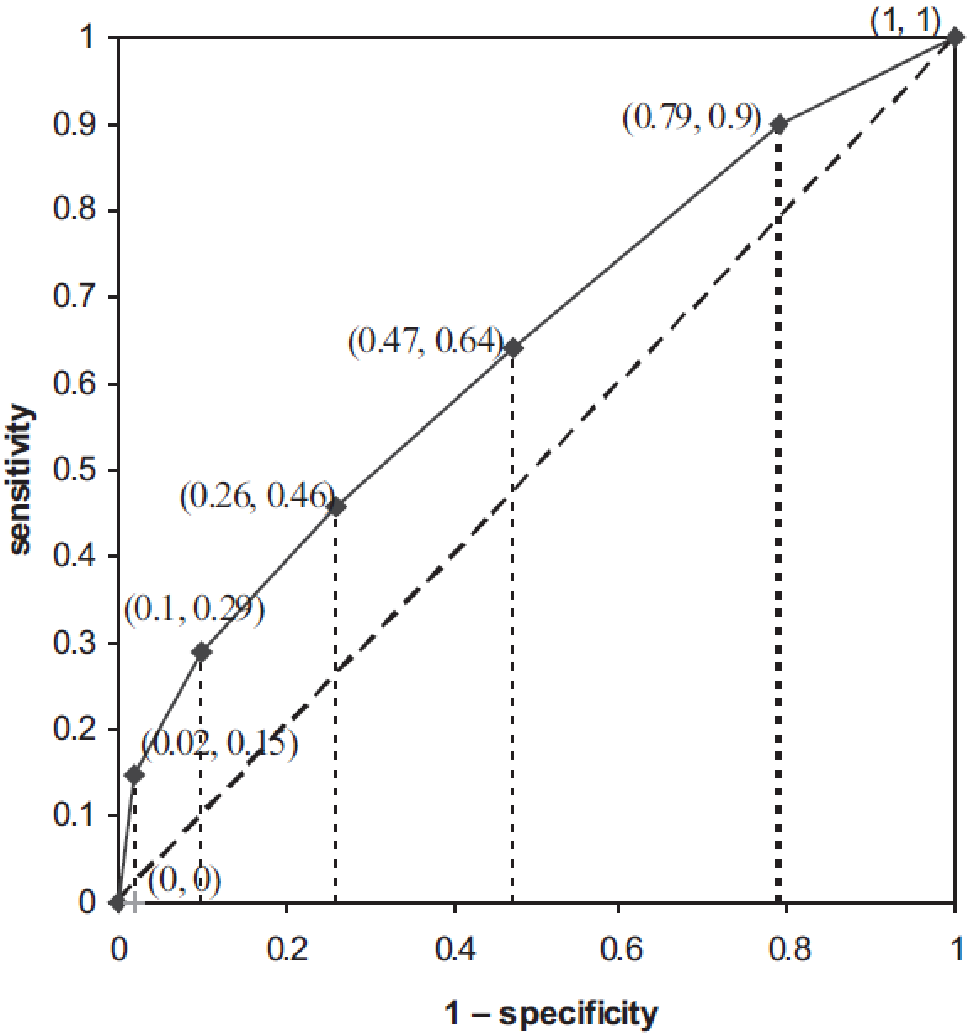
občutljivost

1-specifičnost



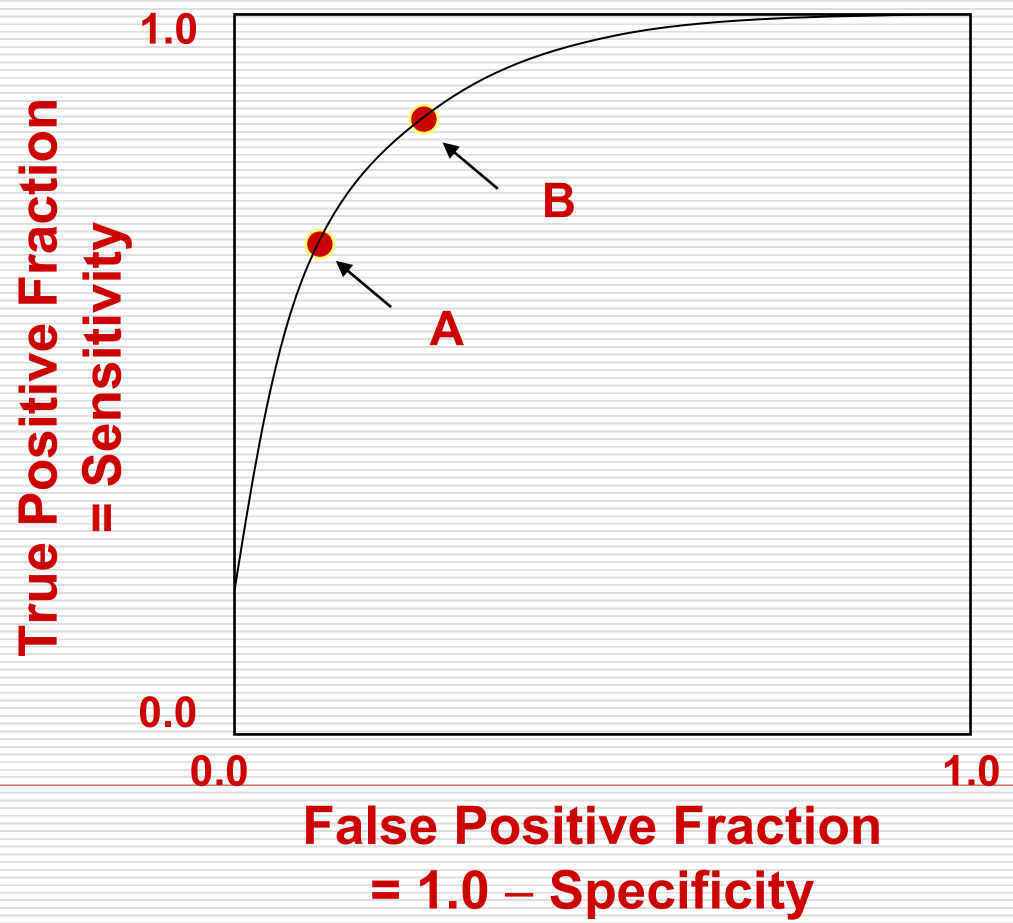
AUROC
Površina pod ROC

AUROC


(1, 1)
(0.79, 0.9)
(0.47, 0.64)
(0.26, 0.46)
(0.1, 0.29)
(0.02, 0.15)
(0, 0)
sensitivity
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0
0.2
0.4
0.6
0.8
1
1 – specificity

Youdenov indeks

- J=občutljivost+specifičnost-1



Tiroksin in hipotiroidizem

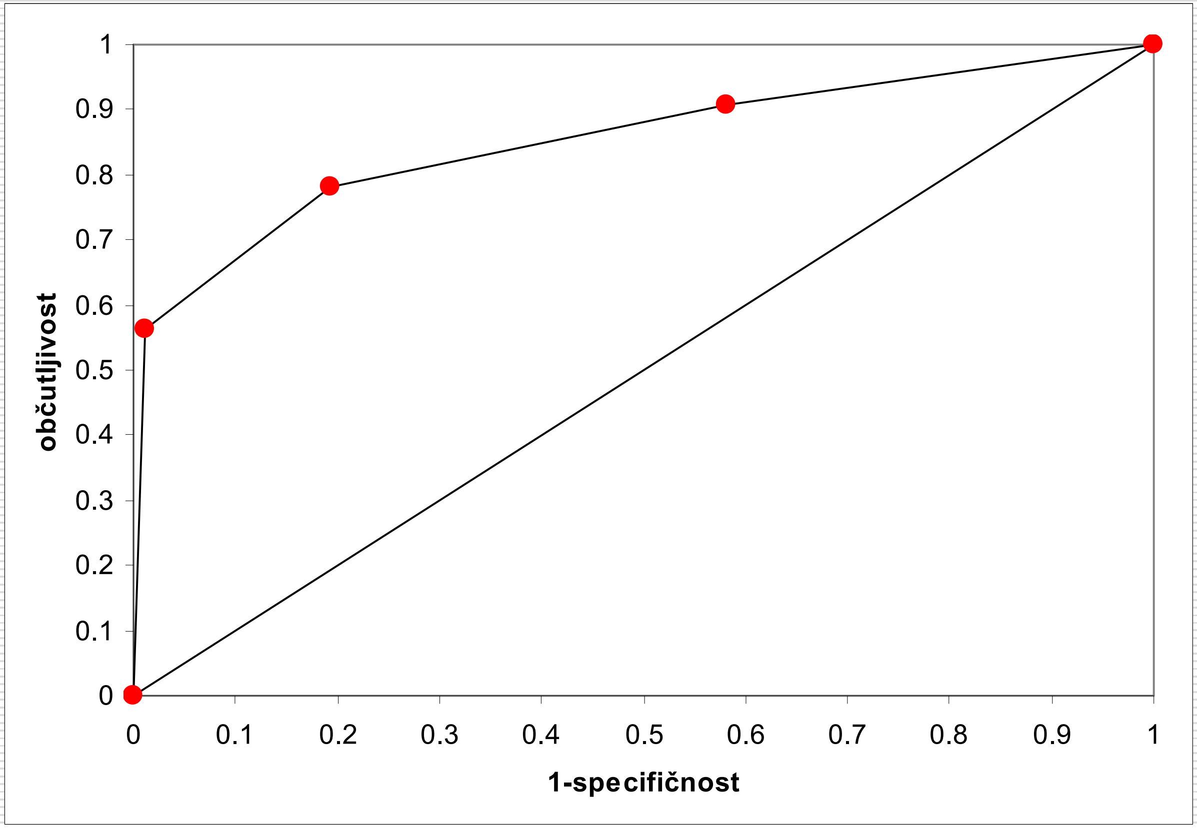
V preglednici so podani rezultati klinične diagnostike in laboratorijske analize tiroksina pri 125 bolnikih s sumom na hipotiroidizem.

Koncentracija T4 (ng/mL)	**Hipotiroidizem**	
	DA	**NE**
≤ 5	18	1
5,1 do 7,0	7	17
7,1 do 9,0	4	36
> 9	3	39

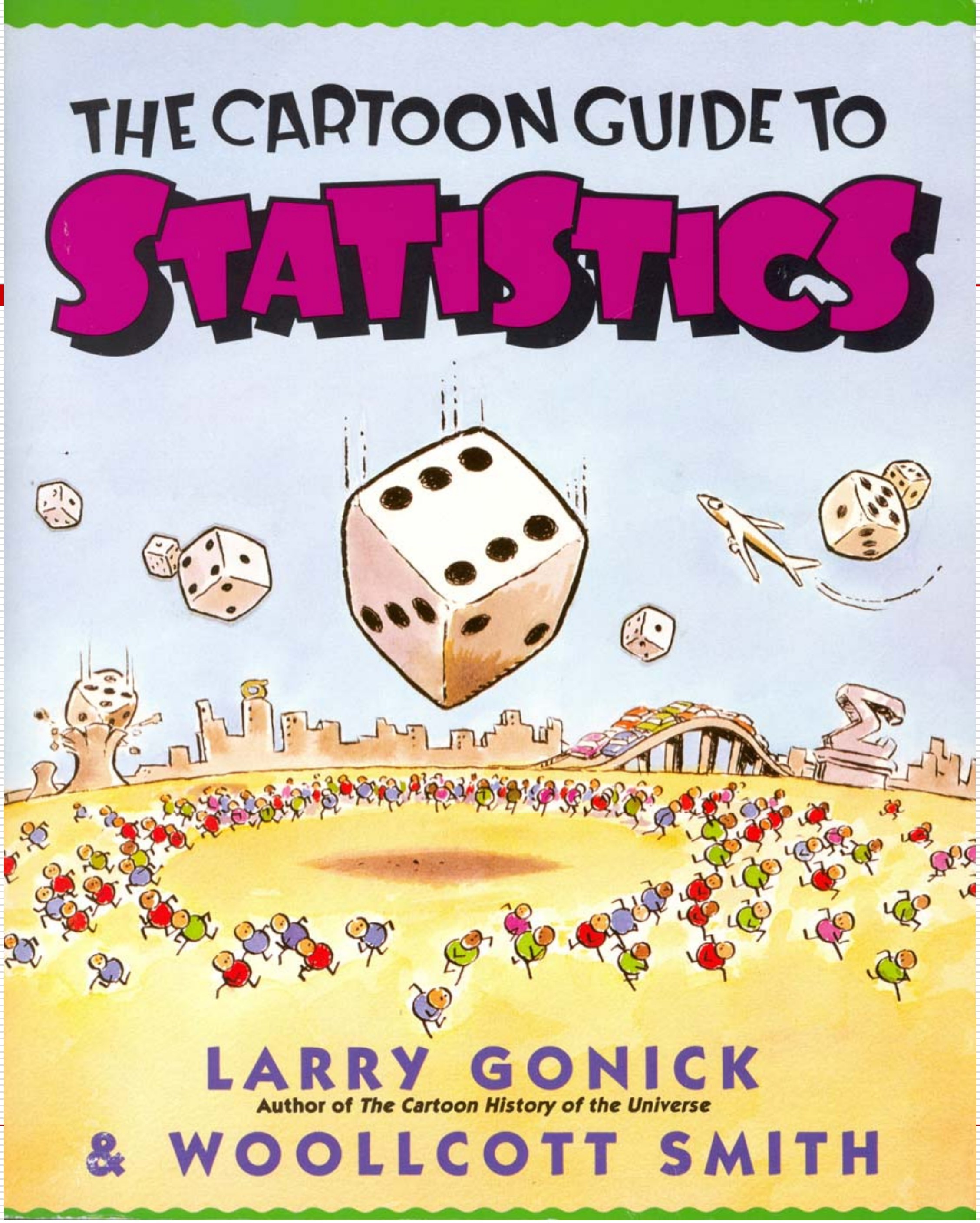
Kakšni sta specifičnost in občutljivost laboratorijskega testa za hipotiroidizem pri mejnih koncentracijah T4 5.0, 7.0 in 9.0 ng/mL?

Skiciraj ROC diagram!

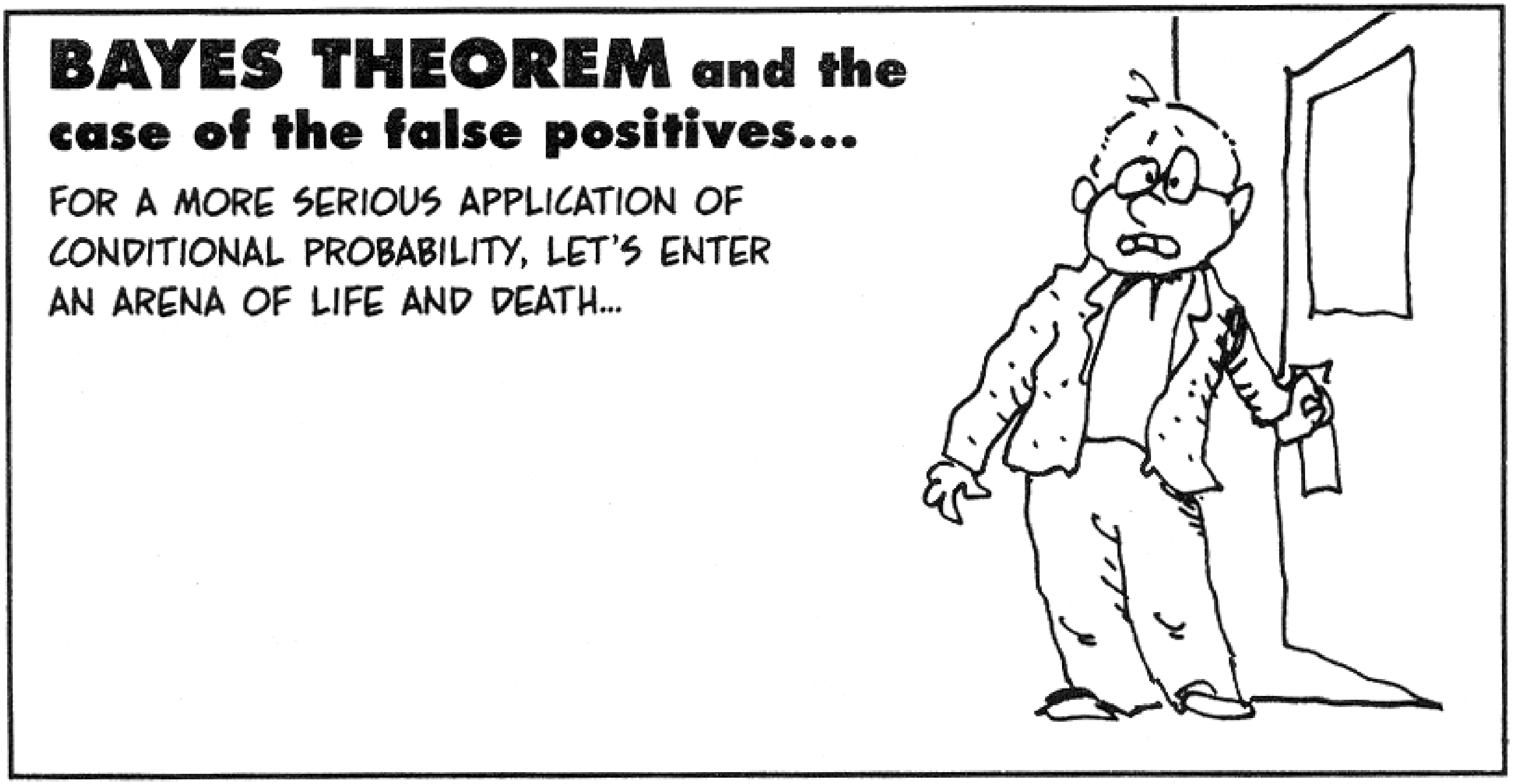
ROC



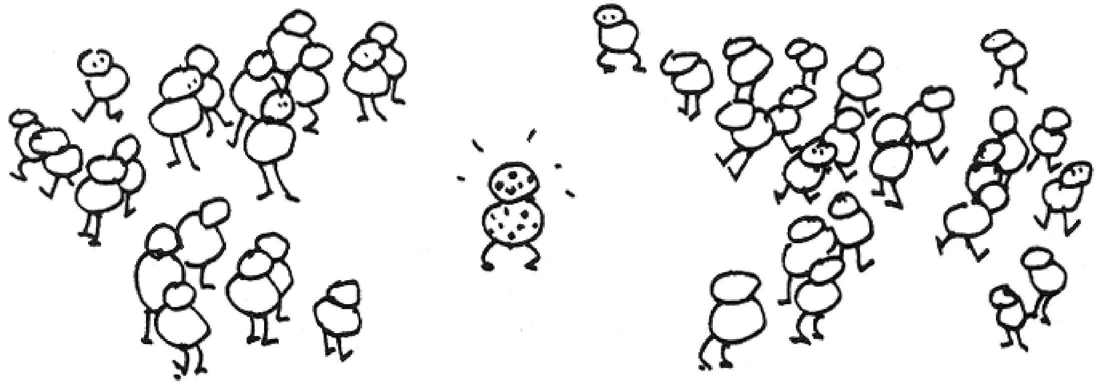
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
občutljivost
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
1-specifičnost


THE CARTOON GUIDE TO
STATISTICS
LARRY GONICK
Author of The Cartoon History of the Universe
& WOOLLCOTT SMITH

Gonick, Smith. Cartoon guide to statistics



SUPPOSE A RARE DISEASE INFECTS ONE OUT OF EVERY 1000 PEOPLE IN A POPULATION...



AND SUPPOSE THAT THERE IS A GOOD, BUT NOT PERFECT, TEST FOR THIS DISEASE: IF A PERSON HAS THE DISEASE, THE TEST COMES BACK POSITIVE ***99%*** OF THE TIME. ON THE OTHER HAND, THE TEST ALSO PRODUCES SOME ***FALSE POSITIVES.*** ABOUT ***2%*** OF UNINFECTED PATIENTS ALSO TEST POSITIVE. AND YOU JUST TESTED POSITIVE. WHAT ARE YOUR CHANCES OF HAVING THE DISEASE?

WE HAVE TWO EVENTS TO WORK WITH:

A : PATIENT HAS THE DISEASE
B : PATIENT TESTS POSITIVE.

THE INFORMATION ABOUT THE TEST'S EFFECTIVENESS CAN BE WRITTEN

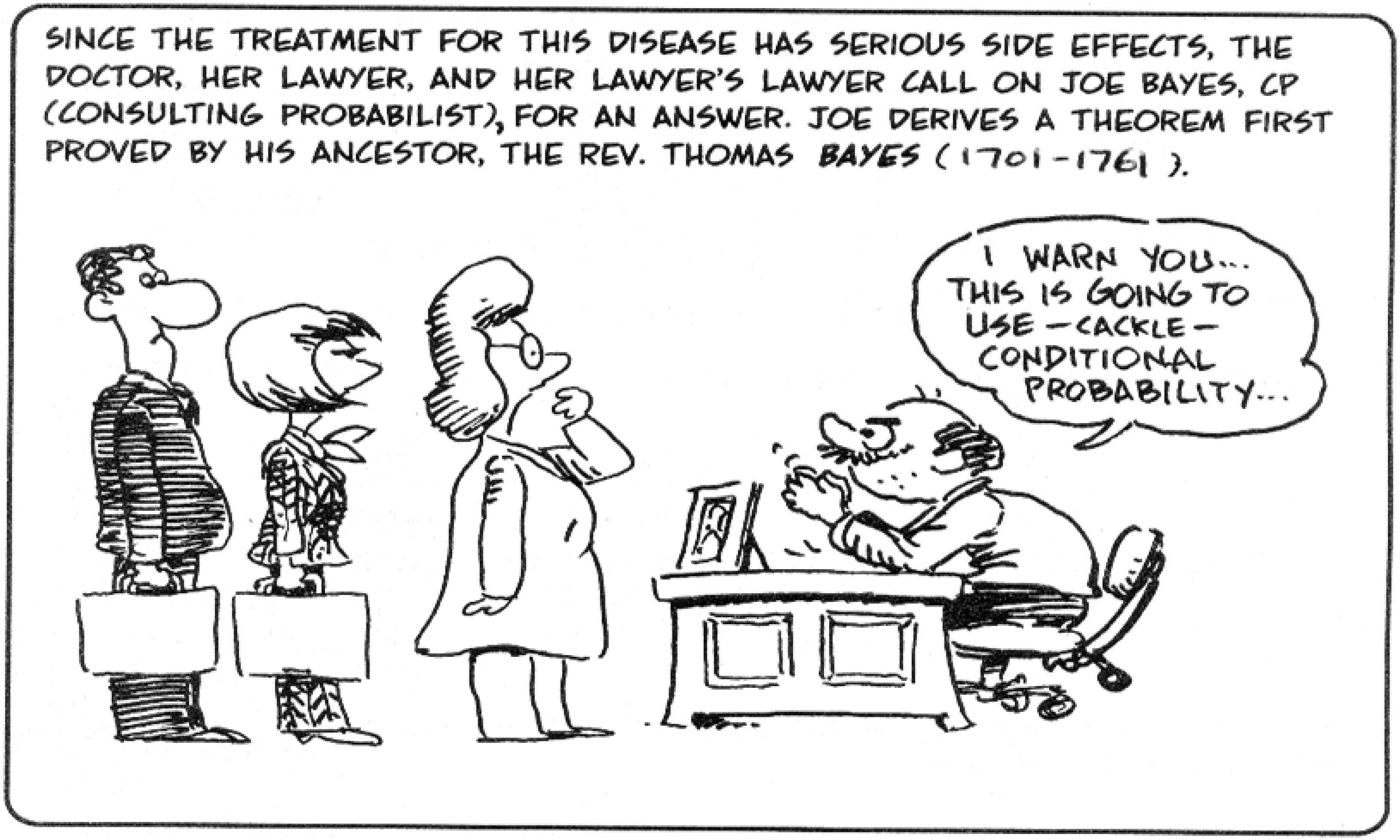
$P(A) = .001$ (ONE PATIENT IN 1000 HAS THE DISEASE)

$P(B|A) = .99$ (PROBABILITY OF A POSITIVE TEST, GIVEN INFECTION, IS .99)

$P(B|\text{NOT } A) = .02$ (PROBABILITY OF A FALSE POSITIVE, GIVEN NO INFECTION, IS .02)

AND WE ASK

$P(A|B) = \text{WHAT?}$ (PROBABILITY OF HAVING THE DISEASE, GIVEN A POSITIVE TEST)


SINCE THE TREATMENT FOR THIS DISEASE HAS SERIOUS SIDE EFFECTS, THE DOCTOR, HER LAWYER, AND HER LAWYER'S LAWYER CALL ON JOE BAYES, CP (CONSULTING PROBABILIST), FOR AN ANSWER. JOE DERIVES A THEOREM FIRST PROVED BY HIS ANCESTOR, THE REV. THOMAS *BAYES* (1701-1761).
I WARN YOU... THIS IS GOING TO USE —CACKLE— CONDITIONAL PROBABILITY...

JOE BEGINS WITH A 2X2 TABLE, WHICH DIVIDES THE SAMPLE SPACE INTO FOUR MUTUALLY EXCLUSIVE EVENTS. IT DISPLAYS EVERY POSSIBLE COMBINATION OF DISEASE STATE AND TEST RESULT.

	A	NOT A
B	A AND B	NOT A AND B
NOT B	A AND NOT B	NOT A AND NOT B

LET'S FIND THE PROBABILITIES OF EACH EVENT IN THE TABLE:

	A	NOT A	SUM
B	P(A AND B)	P(NOT A AND B)	P(B)
NOT B	P(A AND NOT B)	P(NOT A AND NOT B)	P(NOT B)
	P(A)	P(NOT A)	1

THE PROBABILITIES IN THE MARGINS ARE FOUND BY SUMMING ACROSS ROWS AND DOWN COLUMNS.

NOW COMPUTE:

$$P(A \text{ AND } B) = P(B|A)P(A) = (.99)(.001) = .00099$$

$$P(\text{NOT } A \text{ AND } B) = P(B|\text{NOT } A)P(\text{NOT } A) = (.02)(.999) = .01998$$

ALLOWING US TO FILL IN SOME ENTRIES:

	A	NOT A	SUM
B	.00099	.01998	.02097
NOT B	P(A AND NOT B)	P(NOT A AND NOT B)	P(NOT B)
	.001	.999	1

WE FIND THE REMAINING PROBABILITIES BY SUBTRACTING IN THE COLUMNS, THEN ADDING ACROSS THE ROWS.

THE FINAL TABLE IS:

	A	NOT A		
B	.00099	.01998	.02097	P(B)
NOT B	.00001	.97902	.97903	P(NOT B)
	.001	.999	1	
	P(A)	P(NOT A)		

FROM WHICH WE DIRECTLY DERIVE

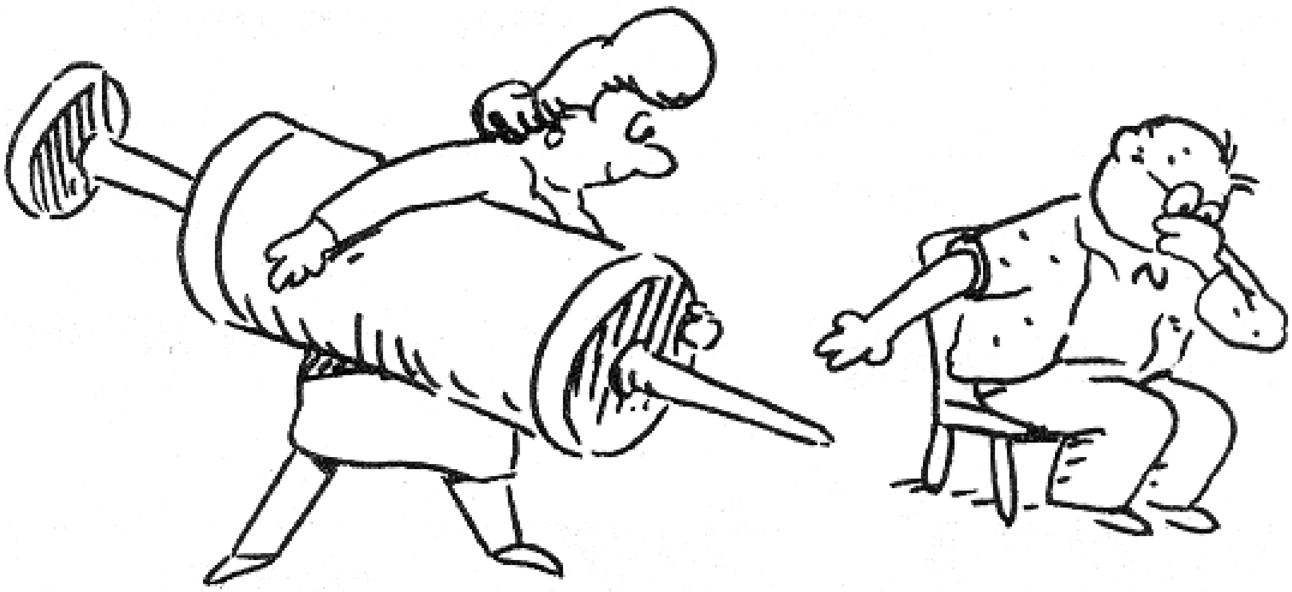
$$P(A|B) = \frac{P(A \text{ AND } B)}{P(B)} = \frac{.00099}{.02097} = .0472$$

DESPITE THE HIGH ACCURACY OF THE TEST, *LESS THAN 5%* OF THOSE WHO TEST POSITIVE ACTUALLY HAVE THE DISEASE! THIS IS CALLED THE *FALSE POSITIVE PARADOX.*

THIS TABLE SHOWS WHAT HAPPENS IN A GROUP OF A THOUSAND PATIENTS. ON AVERAGE, ONLY 21 PEOPLE WILL TEST POSITIVE—AND ONLY *ONE* OF THEM HAS THE DISEASE! 20 FALSE POSITIVES COME FROM THE *MUCH LARGER UNINFECTED GROUP.*

	DISEASE	NO DISEASE	
TESTS POSITIVE	1	20	21
TESTS NEGATIVE	0	979	979
	1	999	1000

WHAT'S THE PHYSICIAN TO DO? JOE BAYES ADVISES HER NOT TO START TREATMENT ON THE BASIS OF THIS TEST ALONE. THE TEST DOES PROVIDE INFORMATION, HOWEVER: WITH A POSITIVE TEST THE PATIENT'S CHANCE OF HAVING THE DISEASE INCREASED FROM 1 IN 1000 TO 1 IN 23. THE DOCTOR FOLLOWS UP WITH MORE TESTS.



Tiroksin in hipotiroidizem

Z uporabo Bayesove formule izračunaj kakšna je pri naključno izbranem pacientu verjetnost hipotiroidizma, če je izid testa pozitiven in kot mejno vrednost izberemo 5.0, 7.0 in 9.0 ng/mL. Pri tem upoštevaj epidemiološke podatke, ki kažejo, da ima hipotiroidizem 0.25% prebivalcev?

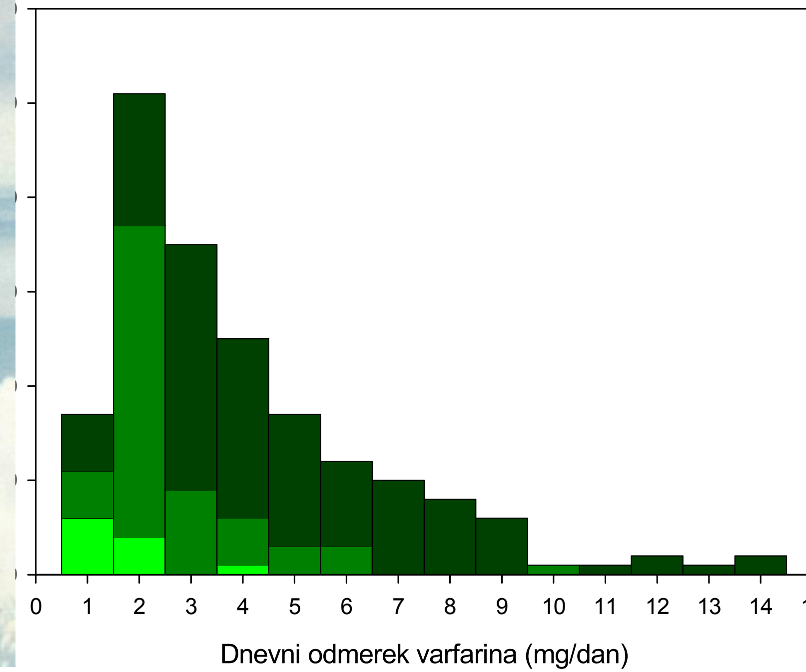
- PNV se s povečevanjem prevalence povečuje.
- Odvisna od občutljivosti in specifičnosti diagnostičnega testa.

Bayesovo adaptivno vodenje

Terapevtski cilj

Predhodno vedenje p(D)

Opazovanje



odmerjanja

Thomas Bayes (1702-1761)

Primer

Kri v blatu pri bolnikih z rakom debelega črevesja

Rezultat endoskopije

Kri v blatu	+	-	
+	20	180	**200**
-	10	1820	**1830**
	30	**2000**	**2030**

Občutljivost, specifičnost, PNV, NNV